15/02/2022: Classical Springer Theory

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0. $\mathfrak{sl}_n$

Take $x \in \mathfrak{sl}_n$, take eigenvalues

$$\phi : \mathfrak{sl}_n \to \mathbb{C}^n / S_n$$

But $\mathrm{Tr}\, x = 0$

$$\Phi : \mathfrak{sl}_n \to \mathbb{C}^{n-1} / S_n$$

where $\mathbb{C}^{n-1} \cong \{ x \in \mathbb{C}^n \mid x_1 + \dots + x_n = 0 \}$

To do this, we look at the space of complete flags

$$\mathcal{B} = \{ 0 = F_0 \subseteq F_1 \subseteq F_2 \subseteq \dots \subseteq F_n = \mathbb{C}^n \}$$

and define the <u>incidence variety</u>

$$\hat{\mathfrak{sl}}_n = \{ (x, F) \in \mathfrak{sl}_n \times \mathcal{B} \mid x(F_i) \subseteq F_i \}$$

By looking at $x|_{F_i / F_{i-1}}$

$$v : \hat{\mathfrak{sl}}_n \to \mathbb{C}^{n-1}$$

$$\begin{tikzcd}
\tilde{\mathcal{N}} \arrow[r, hook] \arrow[d] & \hat{\mathfrak{sl}}_n \arrow[r, "v"] \arrow[d, "\pi"] & \mathbb{C}^{n-1} \arrow[d, "\psi"] \\
\mathcal{N} \arrow[r, hook] & \mathfrak{sl}_n \arrow[r, "\Phi"'] & \mathbb{C}^{n-1}/S_n
\end{tikzcd}$$

If $x \in \mathfrak{sl}_n$ is semisimple and regular, $\pi^{-1}(x)$ has $n!$ points, so we get an action of S_n on it.

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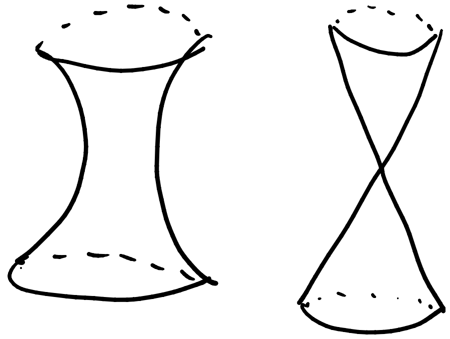
For $\mathfrak{sl}_2$

$$\mathcal{N} = \left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \middle| \, -a^2 + bc = 0 \right\}$$

In general

$$\tilde{\mathcal{N}} \cong T^* \mathcal{B}$$



1. The Grothendieck-Springer simultaneous resolution

- $\mathfrak{sl}_n \to \mathfrak{g}$
- $\mathcal{N} \to \mathcal{N}$ ($x \in \mathfrak{g}$ is nilpotent if ad x)
- $\mathbb{C}^{n-1} \to \mathfrak{h}$
- $S_n \to W$
- $\mathcal{B} \to$ associated flag variety classifies Borel subalgebras (G/B)

Defining

$$\hat{\mathfrak{g}} := \{ (x, b) \in \mathfrak{g} \times \mathcal{B} \mid x \in b \}$$

$$\begin{tikzcd}
\tilde{\mathcal{N}} \arrow[r, hook] \arrow[d] & \hat{\mathfrak{g}} \arrow[r, "v"] \arrow[d, "\mu"'] & \mathfrak{h} \arrow[d, "\pi"] \\
\mathcal{N} \arrow[r, hook] & \mathfrak{g} \arrow[r, "\rho"'] & \mathfrak{h}/W
\end{tikzcd}$$

- How do we define v?

..

- How do we define ν?

For any pair of Borel algebras $\mathfrak{b}, \mathfrak{b}'$, there is a canonical isomorphism
$$\mathfrak{b}/[\mathfrak{b},\mathfrak{b}] \cong \mathfrak{b}'/[\mathfrak{b}',\mathfrak{b}']$$
so we define
$$\nu : \hat{\mathfrak{g}} \longrightarrow \mathfrak{h} : (x, \mathfrak{b}) \mapsto x \bmod [\mathfrak{b},\mathfrak{b}]$$

- How do we define ρ?

We know that $\mathfrak{h}/W = \operatorname{Spec} \mathbb{C}[\mathfrak{h}]^W$, and we have an isomorphism
$$\mathbb{C}[\mathfrak{h}]^W \cong \mathbb{C}[\mathfrak{g}]^G$$
This gives
$$\mathbb{C}[\mathfrak{h}]^W \xrightarrow{\sim} \mathbb{C}[\mathfrak{g}]^G \hookrightarrow \mathbb{C}[\mathfrak{g}]$$
$$\rho : \mathfrak{g} \longrightarrow (\mathfrak{g}/G) \longrightarrow \mathfrak{h}/W$$

$$\begin{tikzcd}
\hat{\mathcal{N}} \arrow[r, hook] \arrow[d, "\mu"'] & \hat{\mathfrak{g}} \arrow[r, "\nu"] \arrow[d, "\mu"'] & \mathfrak{h} \arrow[d, "\pi"] \\
\mathcal{N} \arrow[r, hook] & \mathfrak{g} \arrow[r, "\rho"'] & \mathfrak{h}/W
\end{tikzcd}$$

We are interested in the fibers of $\mu : \hat{\mathcal{N}} \to \mathcal{N}$, the <u>Springer fibers</u>

- If $\mathcal{B}_e := \mu^{-1}(e)$, then
$$\dim \mathcal{B}_e = \frac{1}{2}(\dim \mathcal{N} - \dim G\cdot e) = \frac{1}{2}(\dim C_G(e) - r)$$

- It always holds that
$$\hat{\mathcal{N}} \cong T^*\mathcal{B}$$

<u>Examples</u>

1) $e = 0$, $\mathcal{B}_e = \mathcal{B}$

Examples

1) $e = 0$, $\mathcal{B}_e = \mathcal{B}$

2) e regular nilpotent element ($\dim C_G(e) = r$)

$\hookrightarrow \mathcal{B}_e = \{pt\}$

→ these form one open G-orbit

→ $\pi : \tilde{\mathcal{N}} \to \mathcal{N}$ is a resolution of singularities

3) $\mathfrak{g} = \mathfrak{sl}_3$, $e = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ for a basis $\langle v_1, v_2, v_3 \rangle$

. $0 \subset V_1 \subset \langle v_1, v_2 \rangle \subset V \to \mathbb{P}_1$

. $0 \subset \langle v_1 \rangle \subset V_2 \subset V \to \mathbb{P}_1$

→ this is a union of two $\mathbb{P}^1$'s

This is a general phenomenon!

You can show there is a unique orbit of codim 2 in $\mathcal{N}$, subregular orbit

→ $\mathcal{B}_e$ = union of $\mathbb{P}^1$'s

If $\mathfrak{g}$ is simply laced ones, intersection graph of $\mathcal{B}_e$ is the Dynkin diagram

2. The action of W

2.1 Convolution in homology

Take M_1, M_2, M_3 and $d = \dim M_2$

$$Z_{12} \subset M_1 \times M_2 \quad , \quad Z_{23} \subset M_2 \times M_3$$

and define

$$Z_{12} \circ Z_{23} = \{ (m_1, m_3) \in M_1 \times M_3 \,/\, \exists m_2 \in M_2 \ (m_1, m_2) \in Z_{12} \ (m_2, m_3) \in Z_{23} \}$$

We define the convolution by

$$H_i(Z_{12}) \times H_j(Z_{23}) \to H_{i+j-d}(Z_{12} \circ Z_{23}) :$$

$$(c_{12}, c_{23}) \longmapsto (p_{13})_* (p_{12}^* c_{12} \cap p_{23}^* c_{23})$$

where

$$p_{ij} : M_1 \times M_2 \times M_3 \to M_i \times M_j$$

is the natural projection.

Case of Interest: Let M be a smooth complex manifold, N = variety, we have a proper map $\pi : M \to N$

1) $M_1 = M_2 = M_3 = M$ and $Z = M \times_N M$. The convolution product gives

$$H_\bullet(Z) \times H_\bullet(Z) \to H_\bullet(Z)$$

Prop: $H_\bullet(Z)$ has a natural structure of an associative algebra with unit. If $m = \dim M$, then $H_m(Z)$ is a subalgebra of $H_\bullet(Z)$.

2) Let $x \in N$ and $M_x = \pi^{-1}(x)$. Set $M_1 = M_2 = M$ and $M_3 = \{pt\}$.

Take $Z_{12} = Z$ $\qquad\qquad Z_{23} = M_x$

Then

$$Z \circ M_x = M_x$$

Prop: $H_\bullet(M_x)$ has the structure of a left $H_\bullet(Z)$-module

2.2 The action of W

We define the Steinberg variety

$$Z := \hat{\mathcal{N}} \times_{\mathcal{N}} \hat{\mathcal{N}}$$

Theorem: There is a canonical algebra homomorphism

$$H_m(Z) \simeq \mathbb{Q}[W] \qquad (m = \dim Z)$$

Idea:

1) Z is union of irreducible components, conormal bundles Y_w to G-orbits in $\mathcal{B} \times \mathcal{B}$ indexed by W

2) These form a basis of $H_m(Z)$

3) For $h \in \mathfrak{h}$ a regular semisimple element and $w \in W$, the inverse images under π, $\tilde{\mathfrak{g}}^h$ and $\tilde{\mathfrak{g}}^{w(h)}$ are connected components of the inverse image of the orbit of h. On regular, semisimple elements of $\tilde{\mathfrak{g}}$ we do have a W-action, and this sends $\tilde{\mathfrak{g}}^h$ isomorphically onto $\tilde{\mathfrak{g}}^{w(h)}$. Let

[illegible] do have a W-action, and this sends $\tilde{\mathfrak{g}}^h$ isomorphically onto $\tilde{\mathfrak{g}}^{w(h)}$. Let

$$\Lambda_w^h \subset \tilde{\mathfrak{g}}^h \times \tilde{\mathfrak{g}}^{w(h)}$$

be its graph. It follows that

$$[\Lambda^h_{yw}] = [\Lambda^h_y] * [\Lambda^h_w].$$

Specialising to $h=0$, we can produce similar classes $[\Lambda^0_w]$ independent of h, and by expanding them in $[Y_w]$, we can see that they form a basis.

We have $W \curvearrowright \mathcal{B}_e$. For $e \in \mathcal{N}$, we

$$C(e) := C_G(e)/C_G^0(e)$$

Have an action of $C(e)$ on $H_\bullet(\mathcal{B}_e)$ commutes with the W-action. Let $C(x)^\vee$ the equivalence classes of irrep. in $H_\bullet(\mathcal{B}_e)\otimes \mathbb{C}$. Then

$$\mathbb{C} \otimes H_\bullet(\mathcal{B}_e) = \bigoplus_{\chi \in C(x)^\vee} \chi \otimes \underbrace{H(\mathcal{B}_x)_\chi}$$

for some W-modules $H(\mathcal{B}_x)_\chi$

Springer Correspondence: The set

$$\{ H_{d(x)}(\mathcal{B}_x)_\chi \mid G\text{-equivalence classes of } x \in \mathcal{N},\ \chi \in C(x)^\vee \}$$

is the collection of iso classes of simple W-modules.

2.3 Examples

- x regular nilpotent
  - $\to$ trivial representation
- $x = 0$ we have an isomorphism

$$H^*(G/B) \cong \mathbb{C}[\mathfrak{h}] / \mathbb{C}[\mathfrak{h}]^W_+$$

  - $\to$ regular representation
- x subregular

$$H^2(\mathcal{B}_x)^{\text{triv}}$$

trivial representation of $C(x)$

  - $\to$ reflection representation of W

3. Affine Hecke algebras and Equivariant Cohomology / K-theory

3. Affine Hecke algebras and Equivariant Cohomology / K-theory

3.1 Hecke algebras

→ q-deformation of (extended) affine Weyl group $\hookrightarrow W \ltimes P$

$W \ltimes R$

Two versions:

- graded affine Hecke algebra
  → deformation of $S(\mathfrak{h}^*) \otimes \mathbb{C}[W]$
  depend on some parameter c_i
- affine Hecke algebra
  free $\mathbb{Z}[q, q^{-1}]$-module with basis $\{ e^\lambda T_w \mid w \in W, \lambda \in P \}$
  - $(T_s + 1)(T_s - q) = 0$ s simple reflection
  - $\langle \lambda, \alpha_s^\vee \rangle = \begin{cases} 0 & T_s e^\lambda = e^\lambda T_s \\ 1 & T_s e^{s(\lambda)} T_s = q e^\lambda \end{cases}$

3.2 Equivariant cohomology

Lusztig showed that

$$H_\bullet^{G \times \mathbb{C}^*}(Z) \cong \text{graded AHA} \quad (c_i = 2)$$

→ follows that $H_\bullet^{M_e}(\mathcal{B}_e)$ has an action of graded AHA

+ parametrization of all simple modules dependent on (e, σ, ρ)

nilpotent element; semisimple element with $[\sigma, e] = 2 z_0 e$; rep. of finite group

3.3 Equivariant K-theory

We can show that

$$K^{G \times \mathbb{C}^*}(Z) = \text{AHA}$$

⇝ it follows that for M_e = stabilizer of e in $G \times \mathbb{C}^*$, we have an action of affine Hecke algebra on

$$K^{M_e}(\mathcal{B}_e)$$

For elements of P, this is tensoring with corresponding line bundle on $\mathcal{B}$

If $q \in \mathbb{C}^*$, not a root of unity, we again get a classification of simple modules