$$\begin{tikzcd} & \text{Affine springer fibers} \arrow[dl] \arrow[dr] & \\ \text{DAHA reps} \arrow[rr, leftrightarrow] & & \text{Hilbert schemes} \end{tikzcd}$$

1. Classical Springer theory

G – connected reductive grp

$\mathfrak{g}$ – Lie algebra / $k = \mathbb{C}$

$= \mathbb{F}_q$, $q = p^n$

$B, \mathfrak{b}$ – Borel subgroup / subalgebra

$\mathcal{N}$ – variety of nilpotent elements.

$\tilde{\mathcal{N}} \to \mathcal{N}$ – Springer resolution

$$\tilde{\mathcal{N}} = \{ (x, gB) : \underset{gB \in G/B}{x \in \mathcal{N}}, \; x \in \mathrm{Ad}_g \mathfrak{b} \}.$$

$\tilde{\mathcal{N}} \cong T^* G/B$.

<u>Example:</u> $G = SL_n$, G/B – Flag variety

$$G/B = \{ F_0 \subset \ldots \subset F_n, \; \dim F_i = i \}$$

$$\tilde{\mathcal{N}} = \{ (x, F^\bullet), \; \underset{x\text{ - nilpotent}}{x F_i \subset F_i} \}$$

Springer fiber at $e \in \mathcal{N}$ is a fiber of the map $\tilde{\mathcal{N}} \xrightarrow{\pi} \mathcal{N}$ at e, denoted $\mathcal{B}_e$.

<u>Springer</u>: there is an action of the Weyl group on $H^\bullet(\mathcal{B}_e, \mathbb{Q}$ (or $\mathbb{Q}_\ell$))

<u>Remark</u>: this action does not come from the action of W on $\mathcal{B}_e$.

<u>Examples</u>: $e = 0$, $\mathcal{B}_e = G/B$, $H^\bullet(\mathcal{B}_e, \mathbb{Q}) \simeq$
$\simeq R/R_+^W$, where R is a polynomial ring in $\mathrm{rk}\, G$ variables of degree 2, R_+^W is an ideal generated by symmetric pol. of positive degree.

Rep. of W is isomorphic to a regular rep.

<u>Example</u>: e is regular unipotent, $\mathcal{B}_e$ is a point.

W acts as 1-dim representation (sgn or triv, depending on which def. you use).

<u>Example</u>: $G = SL_n$, e - subregular

$\mathcal{B}_e$ - chain of $n{-}1$ $\mathbb{P}^1$'s 

W acts on $H^2(\mathcal{B}_e)$ as a reflection representation.

$\pi : \tilde{\mathcal{N}} \to \mathcal{N}$

<u>Remark</u>: Definition of the action involves including

Springer fibers to a family. More precisely, W acts on a sheaf $\pi_* \underline{\mathbb{Q}}$.

$M = G \times \mathbb{G}_m^{\times} G\mathcal{N}$, w. $\mathbb{G}_m$ acting by dilation.

M acts on $\tilde{\mathcal{N}} \times_{\mathcal{N}} \tilde{\mathcal{N}}$ (known as Steinberg variety) ($\mathbb{G}_m$ acts trivially on G/B)

$K_M(\tilde{\mathcal{N}} \times_{\mathcal{N}} \tilde{\mathcal{N}})$ was shown by Kazhdan-Lusztig, Chriss-Ginzburg to be isomorphic to the (extended) <u>affine Hecke algebra</u>. This is a certain q-deformation of the ring $\mathbb{Q}[W \ltimes P]$, where P is the weight lattice of G. (Talk 2) David Jordan

Here, roughly, W comes from the fact that $\tilde{\mathcal{N}} \times_{\mathcal{N}} \tilde{\mathcal{N}}$ has W irreducible components, and P from (G-eq. line bundles) line bundles on G/B

$\tilde{H}_{aff}$ - ext. affine Hecke algebra - acts on $K_{M_e}(\mathcal{B}_e)$,

M_e stab. of e in M

Equiv. cohomology $H^\bullet_{M_e}(\mathcal{B}_e)$ carry the action of a so-called graded AHA.

Part coming from W is the deformation of the Springer action above.

This is a part of KL-classification of AHA reps. More in talk 3 (Karim)

Affine Springer fibers

Affine Springer fibers are analogues of Springer fibers for loop groups.

$\mathcal{K} = k((t))$, $\mathcal{O} = k[[t]]$ $\qquad (k = \mathbb{C}$, $\overset{?}{=} \mathbb{F}_q)$

$G(\mathcal{K})$ - loop group, ind-scheme $/k$

$\cup$

$G(\mathcal{O})$ scheme $/k$

$$G(\mathcal{K})/G(\mathcal{O}) = Gr_G \text{ - affine Grassmannian}$$

Example: $G = GL_n$

point of Gr_G is a lattice = projective (= free) $\mathcal{O}$-submodule in $\mathcal{K}^n$ of rank n.

$G(\mathcal{O}) \xrightarrow[t\mapsto 0]{ev} G$, $\quad I := ev^{-1}(B)$ - Iwahori subgroup

$Fl_G := G(K)/I$ - affine flag variety.

Example: $G = GL_n$, $G(K)/I$ classifies sequences of lattices

$$\ldots \subset \Lambda_0 \subset \Lambda_1 \subset \ldots$$

$$\dim_k \Lambda_{i+1}/\Lambda_i = 1, \quad \Lambda_i = t\Lambda_{i+n}.$$

Have a map $Fl_G \xrightarrow{G/B} Gr_G$

Consider $\mathfrak{g}(K)$, $G(K) \circlearrowright \mathfrak{g}(K)$ via the adjoint action.

$\gamma \in \mathfrak{g}(K)$ - $\boxed{\text{regular, semisimple}}$

(meaning it is regular, s.s. in $\mathfrak{g}(\bar{K})$)

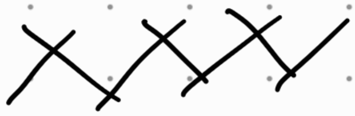
non-reduced

$$\mathcal{X}_\gamma = \{ [g] \in Gr_G, \ Ad(g^{-1})\gamma \in \mathfrak{g}(\mathcal{O}) \}$$

(spherical) affine Springer fiber.

Example: $G = GL_n$, $\mathcal{X}_\gamma = \{$ lattices Λ s.t. $\gamma\Lambda \subset \Lambda \}$

Example: $G = SL_2$, $\gamma = \begin{pmatrix} t & \\ & -t \end{pmatrix}$

$\mathcal{X}_\gamma$ = infinite chain of $\mathbb{P}^1$'s, ...  ...

Similarly, can define (non-spherical) affine

Springer fibers $\mathcal{Y}_\gamma \subset Fl_G$.

We always have a map $\mathcal{Y}_\gamma \to \mathcal{X}_\gamma$.

Affine Springer action

$P^\vee$ - cocharacter lattice
$\tilde{W} = W \ltimes \tilde{P}$ - extended affine Weyl group.

Theorem (Lusztig; Sage):

There is a canonical action of $\tilde{W}$ on $H_\bullet(\mathcal{Y}_\gamma)$

Construction is gluing finite Springer actions.

However, it is not a priori clear how to define an action of $\tilde{W}$ in family (i.e. on some sheaf) since the base is very complicated.

More in Talk 4 (Sarunas)

The analogy of "action in families" was constructed by Yun, using the relation of affine Springer fibers to the Hitchin space (due to Ngô).

This is subject of Talk 5 (Minh-Tâm Trinh)

The action of the affine Weyl group $W \ltimes \underline{P}^\vee$ can be upgraded to an action of a (version) of the double affine Hecke algebra on equivariant homology. (Just as the Springer action of W was upgraded to the action of the affine Hecke algebra on eq. cohomology of Be...
(Varagnolo-Vasserot, Oblomkov-Yun, Yun)
Talks 6, 7 (Lucien, Sasha).
Equivariant homology of AFs, after Goresky-Kottwitz-MacPherson

Relation to Hilbert schemes.

$\mathrm{Hilb}_n = \mathrm{Hilb}_n(\mathbb{C}^2)$ is a scheme parametrizing ideals $I \subset \mathbb{C}[x,y]$ of codimension n, $\dim \mathbb{C}[x,y]/I = n$.

Example: $n=2$

Have ideals $I_{\{p_1, p_2\}}$, $p_1 \in \mathbb{C}^2$, $p_2 \in \mathbb{C}^2$

$$I_{\{p_1,p_2\}} = \{ f \in \mathbb{C}[x,y],\ f(p_1) = f(p_2) = 0 \}$$

and ideals

$$I_{\{p,v\}} = \{ f \in \mathbb{C}[x,y] : f(p) = 0$$

$$df_p(v) = 0$$

v - tangent direction)

$$\mathrm{Hilb}_n \longrightarrow (\mathbb{C}^2)^n / S_n$$ – Hilbert-Chow morphism.

$$\begin{tikzcd} X_n \arrow[r] \arrow[d, "p"'] & (\mathbb{C}^2)^n \arrow[d] \\ \mathrm{Hilb}_n \arrow[r] & (\mathbb{C}^2)^n / S_n \end{tikzcd}$$

X_n - isospectral Hilbert scheme.

Theorem (Haiman)

$p_* \mathcal{O}_{X_n}$ is a vector bundle of rank $n!$ on Hilb_n (called the Procesi bundle), P_n.

We have

- $\mathrm{End}(P_n) = \mathbb{C}[\underline{x}, \underline{y}] \rtimes W$
- $\mathrm{Ext}^i(P_n, P_n) = 0, \; i > 0$

• $P_n^{S_n} \simeq \mathcal{O}_{\mathrm{Hilb}_n}$.

<u>Theorem</u> (Bridgeland - King - Reid)

$R\mathrm{Hom}(P_n, -) : D^b \mathrm{Coh}\, \mathrm{Hilb}_n \simeq$

Talk 8 (Shivang) $\simeq D^b(\mathbb{C}[\underline{x}, \underline{y}] \rtimes W)\text{-mod}$

$\mathbb{C}[\underline{x}, \underline{y}] \rtimes W$ quantizes to

<u>rational Cherednik algebra</u>

(certain degeneration of the DAHA).

Gordon - Stafford: to a filtered module over the <u>spherical</u> subalgebra of RCA associate a coherent sheaf on Hilb_n.

Bezrukavnikov - Finkelberg - Ginzburg: in char p, produce an Azumaya algebra $\mathcal{H}$ on $\mathrm{Hilb}_n^{(1)}$ s.t. $\Gamma(\mathcal{H}) \simeq RCA$

$\mathcal{H}$ splits on the fibers of Hilbert-Chow morphism, giving a derived equiv. of $\mathcal{H}_\xi$ - mod and $\mathrm{Coh}(\mathrm{Hilb}_\xi^{(1)})$

This talk is free!

$G = GL_n$

Finally, let $\gamma_t = z t^d$, z -regular s.s. elt. of $\mathfrak{gl}_n(\mathbb{C})$. Assume $z \in \mathrm{Lie}\,T$

Then $T(\mathbb{C})$ stabilizes γ and acts on $\mathfrak{X}_{\gamma_d} =: \mathfrak{X}_d$

$G \quad H_{T,BM}(\mathfrak{X}_\gamma) \circlearrowleft H_T^\bullet(pt)$ (polynomial ring)

$\Lambda = \begin{pmatrix} t^{d_1} & & \\ & \ddots & \\ & & t^{d_n} \end{pmatrix} \in Z_{G(K)}(\gamma)$

Let Δ be the Vandermonde element in $H_T^\bullet(pt)$

$H^T_{\bullet,BM}(\mathfrak{X}_\gamma)$ is a $\mathbb{C}[\Lambda] \otimes \mathbb{C}[t]$-module

Theorem (Kivinen)

There is a family of ideals

$J^d \subset \mathbb{C}[\underline{x}, \underline{y}]$, $\left(\begin{array}{l} H_T^\bullet(pt) = \mathbb{C}[\underline{y}] \\ \mathbb{C}[\Lambda] = \mathbb{C}[x, x^{-1}] \end{array} \right.$

s.t.

(1) $\Delta^d H^T_{\cdot, BM}(\mathcal{X}_d) \simeq J_x^d$ - local. at x.

(2) $J_x^d \simeq H^0(\mathrm{Hilb}_{\mathbb{C}^\times \times \mathbb{C}}, P_n \otimes \mathcal{O}(d))$

That is, $\mathrm{Proj}(\bigoplus_d J_x^d)$ is an isospectral Hilbert scheme for $\mathbb{C}^\times \times \mathbb{C}$.

Talk 10 - Oscar Kivinen about

WIP w. Gorsky, Oblomkov

affine Springer fiber $\rightsquigarrow$ sheaf on Hilb quantizing to a version of Gordon-Stafford