Hilbert schemes of points on $\mathbb{C}^2$

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$\mathbb{C}^2 = \operatorname{Spec} \mathbb{C}[x,y]$

$\operatorname{Hilb}^n \mathbb{C}^2$ — Hilbert scheme of pts on $\mathbb{C}^2$

It parametrizes ideals $I \subset \mathbb{C}[x,y]$ w. $\dim_{\mathbb{C}} \mathbb{C}[x,y]/I = n$.

$W = \{ x^r y^s : r+s \le N \}$, N large

W spans $\mathbb{C}[x,y]/I$ for N large enough, any I

$H_n \longrightarrow Gr(|W|-n, \operatorname{Span}_{\mathbb{C}} W)$ — injective, onto a locally-closed subset of $Gr(|W|-n, \operatorname{Span}_{\mathbb{C}} W)$

Functor of points:

$$\operatorname{Hilb}^n(\mathbb{C}^2)(S) = \left\{ \begin{array}{ccc} Z & \hookrightarrow & \mathbb{C}^2 \times S \\ \downarrow & & \downarrow \\ S & = & S \end{array} \;\middle|\; Z \to S \text{ is finite flat of degree } n \right\}$$

<u>Examples:</u> $\operatorname{Hilb}^1 \mathbb{C}^2 \cong \mathbb{C}^2$

$\operatorname{Hilb}^2 \mathbb{C}^2 = Bl_{\Delta}(\mathbb{C}^2)^2 / S_2$, $\sigma(p,q) = (q,p)$

$I_{a,b} = \{ p : p(a) = p(b) = 0,\ a \neq b \}$, $I_{a,v} = \{ p : p(a) = \partial_v p(a) = 0 \}$, v - tangent vector at a

<u>Remark:</u> Have a map $\operatorname{Hilb}^2 \mathbb{C}^2 \xrightarrow{\pi} (\mathbb{C}^2)^2 / S_2$ — projective

$I_{a,b} \longmapsto [a,b]$ — pt

$I_{a,v} \longmapsto [a,a]$ — $\mathbb{P}^1$

<u>Theorem:</u> $\operatorname{Hilb}^n \mathbb{C}^2$ is smooth, irreducible of dim $2n$. $\omega_{\operatorname{Hilb}^n \mathbb{C}^2}$ is trivial.

Always have a map $\pi : \operatorname{Hilb}^n \mathbb{C}^2 \to \operatorname{Sym}^n \mathbb{C}^2 = (\mathbb{C}^2)^n / S_n = \operatorname{Spec}(\mathbb{C}[x_1,\dots,x_n,y_1,\dots,y_n]^{S_n})$

$\mathbb{C}[x_1,\dots,x_n,y_1,\dots,y_n] = \mathbb{C}[\underline{x},\underline{y}]$

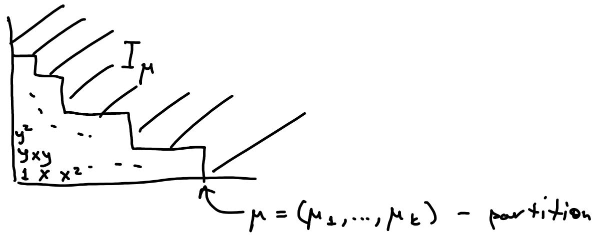
$I \longmapsto \operatorname{Supp} I$ w. multiplicities

<u>Proposition:</u> π is a projective morphism, isomorphism over an open subset. $[p_1,\dots,p_n,\ p_i \neq p_j]$, $p_1 \in \mathbb{C}^2, \dots, p_n \in \mathbb{C}^2$

$T = (\mathbb{C}^\times)^2$ ... on $\mathbb{C}^2$ by scaling the coordinates

$\mathbb{T} = (\mathbb{C}^*)^2$ acts on $\mathbb{C}^2$ by scaling the coordinates.

$\mathbb{T}$ also acts on $\mathrm{Hilb}^n \mathbb{C}^2$. Fixed points are labeled by partitions of n.



- $S_n \circlearrowright \mathbb{C}[\underline{x}, \underline{y}]$ $\quad A = \mathbb{C}[\underline{x}, \underline{y}]^{sgn} = \{ p : \sigma(p) = sgn(\sigma) p \ \ \forall \sigma \in S_n \}$.

  A^k – span of $f_1 \dots f_k$, $f_i \in A$.

  $A^0 = \mathbb{C}[\underline{x}, \underline{y}]^{S_n}$

  $\bigoplus_{i \geq 0} A^i$ – graded ring

Theorem: $\mathrm{Hilb}^n \mathbb{C}^2 \simeq \mathrm{Proj}\left(\bigoplus_{i \geq 0} A^i \right) \xrightarrow{\pi} \mathrm{Spec}(A^0) = \mathrm{Sym}^n \mathbb{C}^2$

Hilbert-Chow morphism

Idea: $D \subset \mathbb{N} \times \mathbb{N}$, $|D| = n$

$D = \{ (p_i, q_j) \}$

$\Delta_D = \det [x_i^{p_j} y_i^{q_j}]_{i,j=1}^n$

We have ~~the~~ open cover of $\mathrm{Hilb}^n \mathbb{C}^2$ by U_μ, μ – partitions

U_μ are ideals I s.t. $\mathbb{C}[x,y]/I$ is spanned by $x^r y^s$, $(r,s) \in \mu$.

$[(x_i, y_i)] \in \mathrm{Sym}^n \mathbb{C}^2$, $\quad (x_i, y_i) \neq (x_j, y_j) \in U_\mu$, for $D = \mu$, $\Delta_\mu(\underline{x}, \underline{y}) \neq 0$.

One can show that $\pi^* \left(\frac{\Delta_D}{\Delta_\mu} \right)$ extends to U_μ.

In fact $U_\mu = \mathrm{Spec}\, \mathbb{C}\left[\frac{\Delta_D}{\Delta_\mu} \right]$

Remark: $\mathrm{Proj}\left(\bigoplus_{i \geq 0} A^i \right) = \mathrm{Proj}\left(\bigoplus_{i \geq 0} A^{2i} \right)$ $\qquad A^2 \subset \mathbb{C}[\underline{x}, \underline{y}]^{S_n} = A^0$. $\mathrm{Hilb}^n \mathbb{C}^2$ as a blow-up of $\mathrm{Sym}^n \mathbb{C}^2$.
ideal

- Isospectral Hilbert scheme X_n

$$\begin{tikzcd} X_n \arrow[r] \arrow[d, "\text{reduced}"] & (\mathbb{C}^2)^n \arrow[d] \\ \mathrm{Hilb}^n\mathbb{C}^2 \arrow[r, "\pi"'] & \mathrm{Sym}^n\mathbb{C}^2 \end{tikzcd}$$

point of X_n : $(I, P_1, \dots, P_n) : \pi(I) = [P_1, \dots, P_n]$

Write $J = A\,\mathbb{C}[\underline{x}, \underline{y}]$ – ideal in $\mathbb{C}[\underline{x}, \underline{y}]$, $J^0 = \mathbb{C}[\underline{x}, \underline{y}]$

Proposition: $X_n = \mathrm{Proj}\left(\bigoplus_{i \geq 0} J^i\right)$.

Theorem (Haiman): X_n is normal, Cohen-Macaulay, Gorenstein.

$\overset{\text{def}}{=}$ every local ring has depth = dim = $2n$

depth – length of maximal regular sequence

Important technical step:

Proposition: J^d is free over $\mathbb{C}[\underline{y}]$ for all d $(X_n \longrightarrow \mathbb{A}^n = \mathrm{Spec}\,\mathbb{C}[\underline{y}])$

Corollary: (of flatness / $\mathbb{C}[\underline{y}]$) $J^d = \bigcap_{i<j} (x_i - x_j, y_i - y_j)^d$

Corollary: (of Theorem)

$X_n \xrightarrow{\alpha} \mathrm{Hilb}^n\mathbb{C}^2$ is flat.

$\alpha_* \mathcal{O}_{X_n} =: P$ – vector bundle of rank $n!$

P – Procesi bundle

Remark: $S_n \circlearrowright X_n$, $S_n \circlearrowright P$

For any $I_\mu \in \mathrm{Hilb}^n\mathbb{C}^2$, $S_n \circlearrowright P_{I_\mu}$ – vector space of dimension $n!$

- fiber P_{I_μ} has an indep. combinatorial description as span of all derivatives of Δ_μ

  The fact that this space has $\dim_{\mathbb{C}} = n!$ is known as "$n!$ – theorem".

- Frobenius character of $P_{I_\mu} \in \Lambda(q,t)$ is known as the modified Macdonald polynomial

  – doubly graded rep. S_n

<u>Affine Springer fibers</u> $G = GL_n$, $\gamma \in \mathfrak{g}(K)$, $\gamma = zt^d$, $z \in T^{rss}(\mathbb{C})$.

$H_*^T(X_\gamma)$ - equivariant BM homology

$X \rightsquigarrow H_*^{BM}(X) = H^{-*}(X, \omega_X)$ $\quad H_T^\bullet(pt) = \mathbb{C}[t]$

$H_*^T(X) = H_{*+2mn}^{BM}(X \times^T ET_m)$ for m large

For an ind-projective scheme $X = \cup X_i$, X_i T-invariant

<u>Define</u>: $H_\bullet^T(X) = \varinjlim H_\bullet^T(X_i)$

<u>Theorem</u> (Localization) Assume X (projective) has f.m. fixed points / T-action.

$\iota_*: H_*^T(X^T) \to H_*^T(X)$

ι_* is an isomorphism after inverting finite number of characters of t.

Assume that T has f.m. 1-dim orbits on X. Then there is a description of $H_*^T(X) \subset H_*^T(X^T) \otimes \mathbb{C}(t)$ in terms of the GKM graph.

<u>Theorem</u> (GKM from Lucien's talk)

T-fixed points are identified w. lattice $\Lambda = P^\vee$, $H^\bullet_{T,ad}(X_\gamma^T) = \mathbb{C}[\Lambda] \otimes Sym(t)$

$$0 \to \sum_{\alpha \in \Phi^+} L_{\alpha,\gamma} \to H^T_{*,ord}(X_\gamma^T) \to H^T_{*,ord}(X_\gamma) \to 0$$

$$L_{\alpha,\gamma} = \sum_{k=1}^{d} (1-\alpha^\vee)^k \mathbb{C}[\Lambda] \otimes Sym(t)\{\partial_\alpha^k\}$$

<u>Theorem</u> (Kivinen) $H_\bullet^T(X_\gamma^T) = \mathbb{C}[\Lambda] \otimes \mathbb{C}[t]$ (x^λ, y)

$J_\alpha^d = \langle y_\alpha, 1 - x^{\alpha^\vee}\rangle^d$, $\quad 1 - x^{\alpha^\vee} \in \mathbb{C}[\Lambda]$, $y_\alpha \in \mathbb{C}[t]$

$\Delta = \prod_{\alpha \in \Phi^+} y_\alpha$

$$\boxed{\Delta^d H_*^T(X_\gamma) = \bigcap_\alpha J_\alpha^d \subset \mathbb{C}[\Lambda] \oplus \mathbb{C}[t]}$$

<u>Corollary</u>: recall $J^d = \bigcap \langle \underset{x_i(1-x^{\alpha^\vee})}{x_i - x_j}, \underset{y_\alpha}{y_i - y_j}\rangle^d$ (remark: equivariant formality

$i<j$

$$\Delta^d H_*^T(X_\gamma) = J_{\underline{x}}^d$$

$$\mathrm{Proj}\Big(\bigoplus_{d\geq 0} \Delta^d H_*^T(X_\gamma)\Big) = X_n(\mathbb{C}^\times \times \mathbb{C})$$

isospectral Hilbert scheme

$$H^0(\mathrm{Hilb}^n(\mathbb{C}^\times \times \mathbb{C}), P \otimes \mathcal{O}(d)) = \Delta^d H_*^T(X_\gamma)$$

$$\gamma = zt^d$$

$\Rightarrow \bigcap_{i<j} \langle x_i - x_j, y_i - y_j \rangle$ is free over $H_T^\bullet(pt) = \mathbb{C}[\underline{y}]$)

$$J^{(d)} = J^d$$

Haiman (uses flatness of J^d) .

<u>BKR theorem:</u>

V - f.d. vector space / $\mathbb{C}$

G - finite group, $G \circlearrowright V$

$V /\!/ G$ - G-Hilbert scheme of V,

$N = |G|$, consider $\mathrm{Hilb}^N(V)$

$$\cup$$

$$\overline{\{\text{free } G\text{-orbits}\}} = V /\!/ G$$

By Haiman's theorem

$$\begin{tikzcd} X_n \arrow[r, hook] \arrow[d] & \mathrm{Hilb}^n \mathbb{C}^2 \times (\mathbb{C}^2)^n \arrow[dl] \\ \mathrm{Hilb}^n \mathbb{C}^2 & \end{tikzcd}$$

By universal properties, have a map $\mathrm{Hilb}^n \mathbb{C}^2 \longrightarrow \mathrm{Hilb}^{n!}(V)$, $V = (\mathbb{C}^2)^n$

<u>Corollary</u> of Haiman's theorem is that $\mathrm{Hilb}^n \mathbb{C}^2 \simeq (\mathbb{C}^2)^n /\!/ S_n$.

$(\mathbb{C}^2)^n /\!/ S_n$ a crepant resolution

Can apply BKR theorem that says $D^b \mathrm{Coh}(V/\!/G) \simeq D^b \mathrm{Coh}^G(V) \simeq D^b \big(G \ltimes \mathbb{C}[V]\text{-modules}\big)$

$$G \ltimes \mathbb{C}[V] = S_n \ltimes \mathbb{C}[\underline{x}, \underline{y}]$$

$$D^b \mathrm{Coh}(\mathrm{Hilb}^n(\mathbb{C}^2)) \longrightarrow D^b(S_n \ltimes \mathbb{C}[\underline{x},\underline{y}]\text{-mod})$$

$$\mathcal{F} \longmapsto R\mathrm{Hom}^\bullet(P, -)$$

P - Procesi bundle

<u>Theorem</u>

- $\mathrm{End}(P) = S_n \ltimes \mathbb{C}[\underline{x}, \underline{y}]$

- $\mathrm{Ext}^i(P,P) = 0$, for $i > 0$.