A TALK ABOUT SOMETHING

19 April 2022 15:37

$\mathbb{C}^\times \times \mathbb{C}$

Goal: R.C.A.'s of type A_{n-1} quantise $\mathrm{Hilb}^n(\mathbb{C}^2)$

Type of Beilinson–Bernstein theorem:

$\mathfrak{g}$ simple L.A. $\quad U_\lambda = \dfrac{\mathcal{U}(\mathfrak{g})}{m_\lambda \mathcal{U}(\mathfrak{g})}$-mod $\xrightarrow{\sim}$ $\mathcal{D}^\lambda_{\mathcal{B}}$-mod $\quad$ for good enough λ

m_λ max^l ideal of $Z(\mathcal{U}(\mathfrak{g})) \simeq S(\mathfrak{g})^G \simeq \mathbb{C}[\mathfrak{h}^*/W]$ $\qquad$ λ-twisted differential operators on $\mathcal{B} = G/B$

Functor are: $\longleftarrow$ global sections

$\longrightarrow$ localisation $\qquad \mathfrak{g} \longrightarrow \mathcal{D}^\lambda_{\mathcal{B}}$ coming from $G \curvearrowright G/B$

Now there are general B-B theorems: world of symplectic resolutions of symplectic singularities [Braden – Licata – Proudfoot – Webster; McGerty – Nevins]

B-B: $\underline{T^*\mathcal{B}} \xrightarrow{\mu} \mathcal{N}$ Springer resolution

$\mathrm{Hilb}^n(\mathbb{C}^2) \longrightarrow \mathrm{Sym}^n(\mathbb{C}^2)$ was first ever case that wasn't a cotangent bundle.

clunky

Noncommutative Algebraic Geometry: $\mathbb{Z}$-algebras

Background: $\quad$ i) $\mathrm{Hilb}^n(\mathbb{C}^2) = \left\{ I \triangleleft \mathbb{C}[x,y] : \dim_{\mathbb{C}} \dfrac{\mathbb{C}[x,y]}{I} = n \right\} = \mathrm{Proj}\left(\bigoplus_{i\geq 0} A^i\right)$

Haiman

$\left(\dfrac{\boxed{S_n \curvearrowright \mathbb{C}^n}}{S_n \curvearrowright \mathfrak{h} \text{ ref}^n \text{ rep}^n}\right)$

$$\downarrow$$

$$\mathrm{Sym}^n(\mathbb{C}^2) = \mathrm{Spec}\left(\mathbb{C}[\underline{x},\underline{y}]^{S_n}\right)$$

where $A^0 = \mathbb{C}[\underline{x},\underline{y}]^{S_n}$, $\quad A^1 = \mathbb{C}[\underline{x},\underline{y}]^{\mathrm{sgn}}$, $\quad A^i = (A^1)^i$

$\mathcal{O}(1) = \Lambda^n \mathrm{Taut}$

$$A^i = H^0(\mathrm{Hilb}^n\mathbb{C}^2, \mathcal{O}(i))$$

Isospectral Hilbert Scheme

Haiman

$$X_n = \left(\mathrm{Hilb}^n(\mathbb{C}^2) \times_{\mathrm{Sym}^n(\mathbb{C}^2)} (\mathbb{C}^2)^n\right)_{\mathrm{red}} = \mathrm{Proj}\left(\bigoplus_{i\geq 0} J^i\right)$$

$$J^1 = A^1\,\mathbb{C}[\underline{x},\underline{y}] \qquad J^i = (J^1)^i = H^0(\mathrm{Hilb}^n(\mathbb{C}^2), \mathcal{P}\otimes \mathcal{O}(i))$$

$\rightsquigarrow$ BKR $\quad D^b(\mathbb{C}[\underline{x},\underline{y}] \rtimes S_n\text{-mod}) \xrightarrow{\sim} D^b(\mathrm{Coh}\;\mathrm{Hilb}^n(\mathbb{C}^2))$

[illegible]

$\rightsquigarrow$ BKR $\quad D^b(\mathbb{C}[x,y] \rtimes S_n\text{-mod}) \xrightarrow{\sim} D^b(\mathrm{Coh}\,\mathrm{Hilb}^n(\mathbb{C}^2))$

ii) R.C.A. of type A : — 1-parameter flat deformation of $\mathrm{Diff}(\mathbb{C}^n) \rtimes S_n$ $\underset{\parallel}{}$ $A_n(\mathbb{C})$

with differential operators have simple poles along reflecting hyperplanes of $S_n \curvearrowright \mathbb{C}^n$: H_c

- x_i $\qquad i = 1, \dots, n$
- $\sigma \in S_n$
- $\underline{\dfrac{\partial}{\partial x_i} - \displaystyle\sum_{j \neq i} \dfrac{c}{x_i - x_j}}\,(1 - (ij))$ $\qquad i = 1, \dots, n$

euler operator
$\deg x_i = 1$
$\deg \sigma = 0$
$\deg \dfrac{\partial}{\partial x_i} = -1$

(lives in $\mathrm{Diff}(\mathbb{C}^n_{reg}) \rtimes S_n$)

$\mathbb{C}^n_{reg} = \mathbb{C}^n \setminus$ ref hyp

$$e = \frac{1}{n!} \sum_{\sigma \in S_n} \sigma \qquad U_c = e H_c e \text{ — spherical rat}^{l}\text{ Ch. alg.}$$

a flat deformation of $e(\mathrm{Diff}(\mathbb{C}^n) \rtimes S_n)e = \mathrm{Diff}(\mathbb{C}^n)^{S_n}$

Behaviours varies with c, but $c \notin (-1,0)$ it's "good enough"

$$U_c\text{-mod} \xrightarrow{\sim} H_c\text{-mod} \quad \text{via } {}_{eH_ce}|eH_c|_{H_c} \quad \text{for good enough } c$$

B-B "translation functors" are important : analogue for RCA "shift functors" — $\delta = \prod_{i<j}(x_i - x_j)$ $\quad (\mathbb{C}[\mathbb{C}^n][\delta^{-1}] = \mathbb{C}[\mathbb{C}^n_{reg}])$

Heckman - Opdam :

$$\delta\, U_c\, \delta^{-1} = U^{-}_{c+1} = e_- H_{c+1} e_-$$

$e_- = \frac{1}{n!} \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma)\sigma$

CRITICAL FACT : $\quad e H_{c+1} e_- \delta = {}_{U_{c+1}}|e H_{c+1} \delta e|_{U_c}$

$$U_c\text{-mod} \rightleftarrows U_{c+1}\text{-mod}$$

$c \geq 0$ equiv

$c \leq -2$ equiv

$\cdot_c \longleftarrow \cdot_{c+1} \cdots$ equiv

$c \leq -2$ equiv

QUANTISATION : natural filtrations on algebras/modules whose associated graded is commutative objects we want

$\mathrm{Diff}(\mathbb{C}^n_{reg}) \rtimes S_n$ differential operator filt^n

$F^0 = \mathbb{C}[\mathbb{C}^n_{reg}] \rtimes S_n \qquad F^i = (F^1)^i$

$F^1 = F^0 + \mathbb{C}[\mathbb{C}^n_{reg}]\left\{\frac{\partial}{\partial x_i}\right\}$

Anything $X \subset \mathrm{Diff}(\mathbb{C}^n_{reg}) \rtimes S_n$ gets induced filt^n

$$F^i X = X \cap F^i$$

$$\mathrm{gr}\ H_c = \mathbb{C}[\underline{x}, \underline{y}] \rtimes S_n$$

$$\mathrm{gr}\ U_c = \mathbb{C}[\underline{x}, \underline{y}]^{S_n} e \cong \mathbb{C}[\underline{x}, \underline{y}]^{S_n} = A^0$$

$$\mathrm{gr}\ eH_c = \mathbb{C}[\underline{x}, \underline{y}]e \cong J^0$$

$$\mathrm{gr}\ eH_c \delta e = \mathbb{C}[\underline{x}, \underline{y}]^{sgn} \delta e \cong A^1, \text{ etc}$$

$\mathbb{Z}$-algebra : Serre's theorem : $X = \mathrm{Proj}(A) \quad A = \bigoplus_{i \geq 0} A^i$ graded f.g.

$$\mathrm{Coh}\, X \xrightarrow{\sim} \mathrm{gr}\ A\text{-mod} / \mathrm{tors}\ A\text{-mod}$$

$$\mathcal{F} \longmapsto \bigoplus_{i \geq 0} H^0(X, \mathcal{F}(i))$$

tors A-mod = full subcat of gr A-mod bounded i.e. $M = \bigoplus M_n$ where $M_n = 0$ for almost all n

We want to use this when

i) $X = \mathrm{Hilb}^n(\mathbb{C}^2) = \mathrm{Proj}(\bigoplus A^i)$

ii) Noncomm. version $\begin{cases} A^0 \rightsquigarrow U_c \\ A^1 \rightsquigarrow eH_{c+1}\delta e \end{cases}$

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A^i consider

$$eH_{c+i}\delta e \underset{U_{c+i-1}}{\otimes} \cdots \underset{U_{c+2}}{\otimes} eH_{c+2}\delta e \underset{U_{c+1}}{\otimes} eH_{c+1}\delta e \xrightarrow[\sim]{\text{mult}} eH_{c+i}\delta eH_{c+i-1}\delta e \cdots eH_{c+2}\delta eH_{c+1}\delta e$$

This is a quantisation of A^i

$$\underset{U_{c+1}}{U_c \oplus eH_{c+1}\delta e} \underset{U_c}{\oplus} \cdots \quad \text{— not algebra}$$

Bondal - Orlov : correct analogue of A is a (lower triangular) $\mathbb{Z}$-algebra

$$\hat{B} = \bigoplus_{i \geq j \geq 0} B_{i,j}$$

- matrix mult^n $B_{i,j} B_{j,k} \subseteq B_{i,k}$
  0 o/w
- $1_i \in B_{i,i}$ local unit

Category of graded modules $M = \bigoplus M_i$

$B_{ij} M_j \subseteq M_i$

& torsion modules $M_n = 0$ for almost all n

$$\mathrm{Coh}\,\hat{B} = \text{gr } \hat{B}\text{-mod} \big/ \text{tors } \hat{B}\text{-mod}$$

e.g. 1 $A = \bigoplus_{i \geq 0} A^i$ set $A_{i,j} = \begin{cases} A^{i-j} & i \geq j \geq 0 \\ 0 & \text{o/w} \end{cases}$

$\hat{A} = \bigoplus A_{i,j}$

$$\mathrm{Coh}\,\hat{A} \underset{\text{triv}}{\hat{=}} \text{gr}A\text{-gmod} \big/ \text{tors } A \cong \mathrm{Coh}\,(X)$$

2. $\hat{B}_c = \bigoplus_{i \geq j \geq 0} B_{i,j}$ where $B_{i,j} = \begin{cases} eH_{c+i}\delta e \cdots eH_{c+j+1}\delta e & i \geq j \geq 0 \\ eH_{c+i}e & i = j \geq 0 \end{cases}$

2. $B_c = \bigoplus_{i \geq j \geq 0} B_{i,j}$ where $B_{i,j} = \begin{cases} eH_{c+i}e & i=j \geq 0 \\ 0 & \text{o/w} \end{cases}$

$\operatorname{gr} \hat{B}_c = \operatorname{gr} \hat{A}$ from Haiman

$\mathbb{Z}$-algebra : organising principle

Each of the B_{ij} : $U_{c+j}\text{-mod} \xrightarrow[\text{Mor}]{\sim} U_{c+i}\text{-mod}$

Morita $\mathbb{Z}$-algebra : $U_c\text{-mod} \xrightarrow{\sim} \operatorname{Coh} \hat{B}_c$ (B-B theorem)

$\operatorname{Coh} \hat{B}_c$ (use filt^{ns}) degenerates to $\operatorname{Hilb}^n(\mathbb{C}^2)$ coherent sheaf

$\uparrow$

U_c-mod (use filt^{ns}) i.e. $\overline{|eH_c|}^{S_n}_{H_c}{}^{S_n} \rightsquigarrow \mathcal{P}$

internal grading

$U_c \;|\;$ projective

$\mathbb{Z}$-graded

bigraded

$S_c : U_c\text{-mod} \xrightarrow{\sim} U_{c+1}\text{-mod}$

Bizarre connection BKR

$\text{filt } H_c\text{-mod} \overset{①}{\underset{②}{\rightrightarrows}} \operatorname{Coh} \operatorname{Hilb}^n(\mathbb{C}^2)$

$(eM, \underline{S_c(eM)}, \underline{S_{c+1}S_c(eM)}, \ldots)$

$\parallel$

① $\underline{\underline{M \in H_c\text{-mod}}} \longmapsto eM \in U_c\text{-mod} \longmapsto \bigoplus_{i \geq 0} B_{i0} \otimes_{U_c} eM$

$\longmapsto \bigoplus_{i \geq 0} \operatorname{gr}(B_{i0} \otimes_{U_c} eM)$

$\longmapsto \mathcal{M} \in \operatorname{Coh} \operatorname{Hilb}^n(\mathbb{C}^2)$

$\parallel ?$

② $M \longmapsto \operatorname{gr} M \in \mathbb{C}[\underline{x}, y] \rtimes S_n\text{-mod} \xrightarrow{\text{BKR}} \Phi(\operatorname{gr} M) \in D^b(\operatorname{Coh} \operatorname{Hilb}^n(\mathbb{C}^2))$