Affine Springer fibres

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Fix notation

G reductive connected gp over K $\quad K = \mathbb{C}$

$\mathbb{F}_q$

$F = K((t))$ $\quad$ $\mathrm{Val} : F^\times \to \mathbb{Z}$

$\mathcal{O}_F = K[[t]]$ $\quad$ $f = \sum_{i=N} \lambda_i t^i \mapsto N$

$N < 0$

Loop groups, affine grassmanians

$LG(R) = G(R((t)))$ Loop Group

R - K-algebra

Not representable by a scheme

$L^+G(R) = G(R[[t]])$

This is a scheme but infinite type

$G = GL_n$

$L^+GL_n(K) = GL_n(K[[t]])$

$L^+ GL_n(k) = GL_n(k[[t]])$

Invertible matrices $A \iff \det A$ is a unit

$\det A \neq 0$ $\quad A \quad a_{ij} = \sum a_{ij}^{k} t^{k}$

an open subscheme inside Infinite affine Space, on coordinates $a_{ij}^{(k)}$

$L\,GL_n$ has an ind scheme structure

$$L\,GL_n = \varinjlim X_m$$

$X_m(R)$ invertible matrices with coefficients in $t^{-m} R[[t]]$

<u>Affine grassmanian:</u>

$$Gr_G(R) = \frac{LG(R)}{L^+G(R)}$$

For GL_n, have interpretation in terms of <u>lattices</u>:

a Lattice Λ is a $R[[t]]$ projective submodule of $R((t))^n$

$$(t^m R[[t]])^n \subset \Lambda \subseteq (t^{-m} R[[t]])^n \quad (1)$$

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$$Gr_{GL_n} = \varinjlim X_m$$

$$X_m = \{ \Lambda \mid \text{condition (1) holds for } m \}$$

$$Gr_{GL_n}(k) = GL_n(k((t))) / GL_n(k[[t]])$$

$$\Lambda_s = \mathcal{O}_F^n \qquad L_n = \{ \text{Lattices in } F^n \}$$

$$GL_n(k((t))) \curvearrowright L_n$$

$$GL_n(k((t))) \to L_n$$

$$g \mapsto g\Lambda_s$$

$$GL_n(k((t))) / GL_n(k[[t]]) \xrightarrow{\sim} L_n$$

SL_n

$$Gr_{SL_n}(k) = \{ \Lambda \text{ lattices} \mid [\Lambda : \mathcal{O}_F^n] = 0 \}$$

$$[\Lambda_1 : \Lambda_2] = \dim_k \Lambda_1 / (\Lambda_1 \cap \Lambda_2) - \dim_k (\Lambda_2 / \Lambda_1 \cap \Lambda_2)$$

Sp_n

$$Gr_{Sp_n}(K) = \{ \Lambda \text{ lattices} \mid \Lambda = \Lambda^* \}$$

Affine springer fibres

$\mathfrak{g}(R) = \mathfrak{g} \otimes_K R$

$\gamma \in \mathfrak{g}(F) = \mathfrak{g} \otimes_K F$

$$\mathcal{X}_\gamma(R) = \left\{ g \in Gr_G^{(R)} \;\middle|\; Ad(g^{-1})\cdot\gamma \in \mathfrak{g}(R[[t]]) \right\}$$

Usually want γ regular semisimple

Prop $\mathcal{X}_\gamma$ is finite dimensional $\iff$ γ is regular semisimple

$\gamma = 0$

$\mathcal{X}_\gamma = Gr_G$

Lattice interpretation

$G = GL_n$

$$\mathcal{X}_\gamma(K) = \{ \Lambda \text{ lattices} \mid \gamma\Lambda \subset \Lambda \}$$

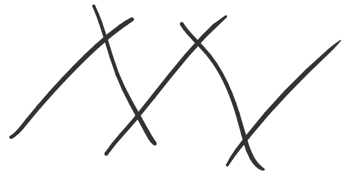
$G = SL_n$

$$\mathcal{X}_\gamma(K) = \left\{ \Lambda \text{ lattices} \;\middle|\; \begin{array}{l} \gamma\Lambda \subset \Lambda \\ [\Lambda, \mathcal{O}_F^n] = 0 \end{array} \right\}$$

Example

$G = SL_2 \qquad \gamma = \begin{pmatrix} t & 0 \\ 0 & -t \end{pmatrix}$

Claim

$\mathcal{H}_\gamma =$ infinite chain of $\mathbb{P}^1$'s 

Find normal form for lattices

$$\Lambda = \begin{pmatrix} v_{11} \\ v_{21} \end{pmatrix} \mathcal{O}_F \oplus \begin{pmatrix} v_{12} \\ v_{22} \end{pmatrix} \mathcal{O}_F$$

$$\Lambda = \begin{pmatrix} t^a \\ c_{b-1} t^{b-1} + \dots c_a t^a \end{pmatrix} \mathcal{O}_F \oplus \begin{pmatrix} 0 \\ t^b \end{pmatrix} \qquad \begin{matrix} a \leq b \\ a, b \in \mathbb{Z} \end{matrix}$$

- $[\Lambda, \mathcal{O}_F^2] = -(a+b)$
- $\gamma \begin{pmatrix} t^a \\ c_{b-1} t^{b-1} + \dots c_a t^a \end{pmatrix} \in \Lambda$

  $c_k = 0 \qquad k < b-1$

$$\Lambda = \begin{pmatrix} t^a \\ c_{b-1} t^{b-1} \end{pmatrix} \mathcal{O}_F \oplus \begin{pmatrix} 0 \\ t^b \end{pmatrix}$$

$[\Lambda, \mathcal{O}_F^2] = 0$

$$[\Lambda, \vartheta_F^L] = 0$$

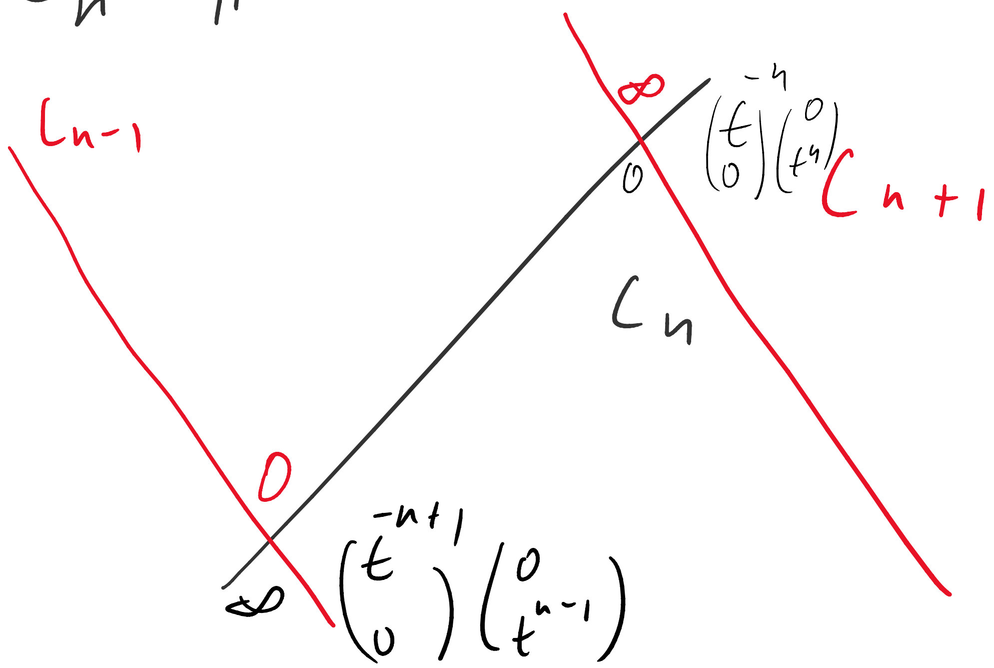
$$a = -b$$

$$\Lambda = \begin{pmatrix} t^{-n} \\ c\, t^{n-1} \end{pmatrix} \vartheta_F \oplus \begin{pmatrix} 0 \\ t^{n} \end{pmatrix}$$

$$C_n = \left\{ \Lambda \;\middle|\; \Lambda \subset \underline{\begin{pmatrix} t^{-n} \\ 0 \end{pmatrix} \vartheta_F \oplus \begin{pmatrix} 0 \\ t^{n-1} \end{pmatrix}} \right\}$$

$$C_n = \left\{ \underset{c \in \mathbb{C}}{\begin{pmatrix} t^{-n} \\ c\, t^{n-1} \end{pmatrix} \vartheta_F \oplus \begin{pmatrix} 0 \\ t^{n} \end{pmatrix} \vartheta_F} \; ; \; \begin{pmatrix} t^{-n+1} \\ 0 \end{pmatrix} \vartheta_F \oplus \begin{pmatrix} 0 \\ t^{n-1} \end{pmatrix} \right.$$

$$C_n \cong \mathbb{P}^1$$



$G = SL_2$

$\gamma = \begin{pmatrix} 0 & t^2 \\ t & 0 \end{pmatrix}$

$\mathcal{X}_\gamma \cong \mathbb{P}^1$

First properties

Prop:

1) γ is conjugate to γ' by $G(F)$ then $\mathcal{X}_\gamma \cong \mathcal{X}_{\gamma'}$

$\gamma = Ad(g)\gamma'$

2) $\mathcal{X}_\gamma$ is non-empty iff $\chi(\gamma) \in c(\mathcal{O}_F)$

$\chi : \mathfrak{g} \longrightarrow c = \mathfrak{g}/\!/G$

Centraliser action

$\gamma \in \mathfrak{g}(F)$

$$\begin{tikzcd} G_\gamma \arrow[r, hook] & G_F \arrow[r] \arrow[d] & G \arrow[d] \\ & \operatorname{Spec} F \arrow[r] & \operatorname{Spec} K \end{tikzcd}$$

centraliser of γ

G_γ is a maximal torus

$LG_\gamma(k) \curvearrowright \mathcal{H}_\gamma$

$h \in G_\gamma(k((t)))$ $\qquad g \in \mathcal{H}_\gamma$

$h \cdot g = hg$

$$\mathrm{Ad}((hg)^{-1})\gamma = \mathrm{Ad}(g^{-1}h^{-1})\gamma = \mathrm{Ad}(g^{-1})\cancel{\mathrm{Ad}(h^{-1})}\gamma$$
$$= \mathrm{Ad}(g^{-1})\gamma + \mathfrak{g}(\mathcal{O}_F)$$

$$X_\bullet(G_\gamma) = \mathrm{Hom}(\mathbb{G}_m(F), G_\gamma(F))$$

Define a lattice Λ_γ

$\lambda \in X_\bullet(G_\gamma)$

$\lambda(t) \in G_\gamma(F)$

$$\Lambda_\gamma = \mathrm{im}(X_\bullet(G_\gamma) \to G_\gamma(F))$$

$$\gamma = \begin{pmatrix} t & 0 \\ 0 & -t \end{pmatrix}$$

$$G_\gamma = \begin{pmatrix} f & \\ & f^{-1} \end{pmatrix} \quad f \in F^\times$$

$$\Lambda_\gamma = \mathbb{Z} \quad \text{generated by } \begin{pmatrix} t & \\ & t^{-1} \end{pmatrix}$$

$\Lambda_\gamma = \mathbb{Z}$ generated by $\left(\; t^{1} \; \right)$

$$\gamma = \begin{pmatrix} 0 & t^n \\ 1 & 0 \end{pmatrix} \quad n \text{ odd}$$

$G_\gamma(F)$ is a non-split torus
only splits over $F(t^{1/2})$

$$\Lambda_\gamma = 0$$

Parahoric springer fibres

G	LG
B borel	I Iwahori $\quad \pi : L^+G \to G$, $t \mapsto 0$ $I = \pi^{-1}(B)$
P parabolic subgroups $B \subset P \subset G$	P Parahoric subgroups $I \subset P \subset LG$ finite codim
(partial) Flag variety $G/B \quad G/P$	(partial) affine Flag variety $LG/I = FL \quad LG/P = FL_P$
D dynkin diagram subsets of D	subsets of $\hat{D}$ Affine dynkin diagram

Subsets of D

$\updownarrow$

parabolics

$B \leftrightarrow \emptyset$

W

Subsets of $\hat{D}$ Affine dynkin diagram

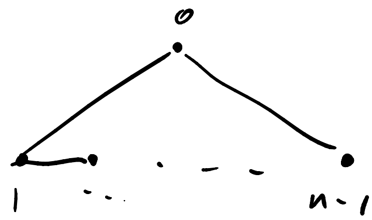
$\updownarrow$

Parahorics

$I \leftrightarrow \emptyset$

$L^+G \leftrightarrow \hat{D} \setminus$ extra pt

$G = SL_n$



Extended affine Weyl gp

$T \subset G$

$\hat{W} = X_\bullet(T) \rtimes W$

If G is simply connected, then $\hat{W}$ is a Coxeter group generated by simple reflections s_i that correspond to the nodes of the affine Dynkin diagram

Parahoric Springer fibres
affine

$$X_{\gamma,P}(k) = \left\{ g \in Fl_P \;\middle|\; Ad(g^{-1})\gamma \in \operatorname{Lie} P \right\}$$

If $P = I$

If $P = I$

$$\mathcal{X}_{\gamma, I} = \mathcal{Y}_\gamma$$

$$P \subset Q$$

Can define projection

$$FL_P \to FL_Q$$

In particular

$$I \subset L^+G$$

$$FL_I \to Gr_G$$

$$\mathcal{Y}_\gamma \to \mathcal{X}_\gamma$$

Lattice interpretation

$G = SL_2$

$$\mathcal{Y}_\gamma = \left\{ \begin{array}{c} \text{chains of lattices} \\ \Lambda_i \end{array} \;\middle|\; \begin{array}{l} \Lambda_i \subset \Lambda_j \quad i \leq j \\ [\Lambda_i, \mathcal{O}_F^2] = i \\ \Lambda_i = t\Lambda_{i+2} \\ \gamma \Lambda_i \subset \Lambda_i \end{array} \right\}$$

$$\Lambda_0 \subset \Lambda_1$$

Example

$$\gamma = \begin{pmatrix} 0 & t^2 \\ t & 0 \end{pmatrix}$$

$\mathcal{Y}_\gamma$ has two irreducible components $\cong \mathbb{P}^1$

$\mathcal{X}_\gamma \cong \mathbb{P}^1$ $\qquad\qquad \mathcal{Y}_\gamma \to \mathcal{X}_\gamma$

Summary of geometric properties of $\mathcal{X}_\gamma$ and $\mathcal{Y}_\gamma$:

1) $\mathcal{X}_\gamma$ is not reduced

2) $\mathcal{X}_\gamma$ is locally of finite type over k

3) There is a projective closed subscheme Z s.t.

$$\mathcal{X}_\gamma^{red} = \bigcup_{l \in \Lambda_\gamma} l Z$$

4) Action of Λ_γ is free and the quotient is a proper algebraic space

5) If G is simply connected then $\mathcal{Y}_\gamma$ is connected and equidimensional

6) dimension formula

$$\dim \mathcal{X}_\gamma = \frac{1}{2}\left(\mathrm{val}(\Delta_\gamma) - c(\gamma) \right)$$

$$c(\gamma) = r - \mathrm{rk}_{\mathbb{Z}} X_*(G_\gamma)$$

Consider adjoint map

$$ad(\gamma) : \mathfrak{g} \longrightarrow \mathfrak{g}$$

$$\overline{ad(\gamma)} : \mathfrak{g}/Z(\mathfrak{g}) \to \mathfrak{g}/Z(\mathfrak{g})$$

$$\Delta(\gamma) = \det \overline{ad}(\gamma)$$

Action of Extended Weyl Group

Thm $\widetilde{W} \curvearrowright H_\bullet(\mathcal{Y}_\gamma)$

Specialise to G simply connected

Sketch of construction :

$$\widetilde{W} = \langle s_0 \quad \cdots \quad s_n \rangle \qquad s_i \longleftrightarrow P_i$$

There is an exact sequence

$$1 \to P_i^+ \to P_i \to L_{P_i} \to 1$$

Levi quotient reductive

We have a Cartesian diagram:

$$\mathcal{Y}_\cdot \xrightarrow{ev} [\tilde{\mathfrak{l}}_{P_i}/\ldots]$$

$\mathfrak{l}_P$ Lie algebra

$$\begin{tikzcd} \mathcal{Y}_\gamma \arrow[r, "ev"] \arrow[d, "\tilde{\pi}"'] & {[\widetilde{\mathfrak{l}}_{P_i}/L_{P_i}]} \arrow[d, "\pi"] \\ \mathcal{X}_{P_i,\gamma} \arrow[r, "ev"] & {[\mathfrak{l}_{P_i}/L_{P_i}]} \end{tikzcd}$$

$\mathfrak{l}_P$ Lie algebra of L_P

$\mathcal{Y}_\gamma = \left\{ g \in FL_I \,\middle|\, \begin{array}{l} Ad(g^{-1})\gamma \\ \in \mathrm{Lie}\, I \end{array} \right\}$

$\mathcal{H}_{P_i,\gamma} \xrightarrow{ev}$

$g \in \mathcal{H}_{P_i,\gamma}$

$Ad(g^{-1})\gamma \in \mathrm{Lie}\, P_i$

$P_i \to L_{P_i}$

$\mathrm{Lie}\, P_i \to \mathrm{Lie}\, \mathfrak{l}_{P_i}$

Project to $\mathrm{Lie}\, \mathfrak{l}_{P_i}$

$\widetilde{\mathfrak{l}}_{P_i} \to \mathfrak{l}_{P_i}$

$\mathfrak{l}_{P_i} = \mathrm{Lie}\, L_{P_i}$

$\widetilde{\mathfrak{l}}_{P_i} = \{ x, B \mid x \in \mathrm{Lie}\, B \quad x \in \mathfrak{l}_{P_i} \}$

For

Classical springer fibres

$W \circlearrowleft R\pi_* \omega$ $\qquad \omega = p^! Q$ $\qquad \pi : \widetilde{\mathfrak{l}}_{P_i} \to \mathfrak{l}_{P_i}$

$W(L_{P_i}) \circlearrowleft R\pi_* \omega$ $\qquad W(L_{P_i}) = \langle s_i \rangle$

Use proper base change

to get action on $R\tilde{\pi}_* \omega$ $\qquad \omega$ on $\mathcal{Y}_\gamma$

to get action on $R\tilde{\pi}_* \omega$ ω on $\mathcal{Y}_\gamma$

therefore, get an action on $H_\bullet(\mathcal{Y}_\gamma)$

Example:

$G = SL_2$

$\gamma = \begin{pmatrix} 0 & t^2 \\ t & 0 \end{pmatrix}$ $\mathcal{Y}_\gamma$ has two irreducible components $[C_0]$ $[C_1]$

can be shown, that

$H_2(\mathcal{Y}_\gamma)$ is a non-trivial extension of the sign representation of $\widetilde{W}$

$$0 \to \mathrm{triv} \to H_2(\mathcal{Y}_\gamma) \to \mathrm{Sgn} \to 0$$