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Ramblings about double affine Hecke algebras

What are Hecke algebras?

1) Algebraic description: deformation $\mathcal{H}^{fin}$ the group algebra $\mathbb{C}[W]$ $(W=S_n)$
2) Convolution algebras on $\mathbb{C}[{}_B\backslash^{G}/_B]$ over $\mathbb{F}_q$
$$\mathcal{H}^{fin} \cong (\mathbb{C}[{}_B\backslash^{G}/_B], *)$$
3) Schur–Weyl duality for $\mathcal{U}_q(\mathfrak{gl}_N)$.

For 1)

$$W = \langle s_i \mid s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1},\ s_i^2=1,\ s_i s_j = s_j s_i \ \ i,j \text{ distant} \rangle$$

$$\mathcal{H}^{fin}_\tau = \mathbb{C}\langle T_i \mid T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1},\ T_i T_j = T_j T_i,\ (T+1)(T-\tau)=0 \ \ ??,\ T^2+(\tau-1)T-\tau=0 \rangle$$

$\tau \in \mathbb{C}$

First appearance (2) $G = SL_n(\mathbb{F}_q)$

$\mathbb{C}[{}_B\backslash^{G}/_B] :=$ $B\times B$-invariant functions on G

$G/_B = \bigcup_{w\in W} \underline{BwB}$ Bruhat decomposition.

$\mathbb{C}[{}_B\backslash^{G}/_B]$ has dim $= |W|$.

$T_w \mapsto$ characteristic function of BwB

$$\begin{array}{c} G \\ \uparrow m \\ G\times G \xrightarrow{\pi_1} G \\ \downarrow \pi_2 \\ G \end{array}$$

$\boxed{\tau = q}$ ← integer

$\mathcal{H}^{fin}_\tau \longrightarrow (\mathbb{C}[{}_B\backslash^{G}/_B], *)$

$$f_1 * f_2(g) = \frac{1}{|B|}\sum_{x\in G} f_1(x)\cdot f_2(x^{-1}g)$$

$$\{T_i^2 = (\tau-1)T_i + \tau\} \longleftrightarrow \underbrace{BwB}_{T_w}\cdot \underbrace{Bs_iB}_{\times T_s} \subseteq BwsB \cup BwB$$
$$= \#\,T_{ws} + \#\,T_w$$

$\dim \mathcal{H}^{fin}_\tau = n! = \dim \mathbb{C}[S_n]$

Aside:
$\mathcal{H}^{fin}_\tau$ had generators $T_1 \dots T_{n-1}$
$w\in W$ $w = s_{i_1}\cdots s_{i_k}$
$\underline{T_w} := T_{i_1}\cdots T_{i_k}$
Braid relations ⇒ independent of reduced expression.

(3) Appearance of $\mathcal{H}^{fin}_\tau$ (only in type A) (other types B, C do appear but using symmetric pairs)

$V =$ defining rep of $GL(n)$

$$V^{\otimes n} = V \otimes \cdots \otimes V, \quad s_i = \underset{i\ \ i+1}{X}$$

$\mathrm{End}_{GL(n)}(V^{\otimes n}) \cong \mathbb{C}[S_n]$ Schur–Weyl duality.

$\mathrm{Rep}\,GL(n) \overset{\text{deformation}}{\rightsquigarrow} \mathrm{Rep}_q GL(n)$
$= $ finite dim'l representations of $\mathcal{U}_q(\mathfrak{gl}_n)$ $\quad V\otimes V = S^2_q(V) \oplus \Lambda^2_q(V)$

$V = \mathbb{C}^n \circlearrowleft \mathcal{U}_q(\mathfrak{gl}_n)$

$V\otimes \cdots \otimes V$, $\bar{s}_i = X$ $\quad s_i \notin \mathrm{End}_{\mathcal{U}_q(\mathfrak{gl}_n)}(V^{\otimes n})$

$T_i \in \mathrm{End}(V\otimes V)$ $\quad (i,\ i+1)$

$T_i = s_i \cdot R_{V,V}$ ← univ. R-matrix.

Martin-Samelson

AG: Elliptic Hall algebra $E_\tau(\mathbb{F}_q)$

Top: Braids, or skeins on T^2

$(\mathrm{ch}(E_\tau(\mathbb{F}_q))$ $\quad$ Van Wasse Schiffmann–Vasserot

sph DAHA

DAHA ⇝ affine Springer theory.

$\mathcal{Y}$ comm. subalgebra; eigenvectors for $\mathcal{Y}$

combinatorics + rep th. periodic Young diagrams symmetric functions (MacDonald conj)

degeneration

trigonometric + rational degenerations

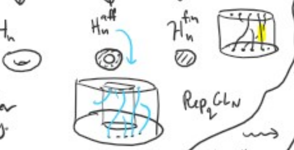
MP/RT commuting variety, eq. $\mathrm{Loc}_{GL_n}(T^2)$ $*$

abelian varieties of $\dim_{\mathbb{C}} = 1$ $\quad E_\tau$ DAHA; $\mathbb{C}^\times$ trig. Deg.; $\mathbb{C}$ rat. Deg.

$\mathbb{C}[B_n(T^2)]$ $\quad \mathbb{C}[B_n(\mathrm{Ann})]$ $\quad \mathbb{C}[B_n]$

$\downarrow \qquad \downarrow \qquad \downarrow$

$\mathbb{H}_n \qquad H_n^{aff} \qquad \mathcal{H}_n^{fin}$



$\mathrm{Rep}_q GL_N$

Commuting variety for GL_n
$= \{(A,B)\in GL_n\times GL_n \ \ AB=BA\}/_{\text{conj.}}$
$\{\pi_1(T^2)\to GL_n\}/_{\text{conj.}}$

"$\simeq$" $(T\times T)/_W$ ← $QC(T\times T/W)$

$(\mathbb{C}[T]\otimes\mathbb{C}[T])^W$ $\quad$ ~~Morita equivalent~~

Chevalley theorem ⇝ $\mathbb{C}[T]\otimes\mathbb{C}[T]\#\mathbb{C}[W]$

$$\boxed{\mathbb{C}[X_1^{\pm1},\dots,X_n^{\pm1}]\otimes_q \mathbb{C}[Y_1^{\pm1},\dots,Y_n^{\pm1}]\otimes \mathbb{C}_\tau[W]}$$

$$w f(x's, y's) w^{-1} = f(w\cdot x's, w\cdot y's)$$

$(\mathrm{comm} \subset G\times G)/G$

$\mathrm{Comm}/G \xrightarrow{\text{global sections}} T\times T/W \xleftarrow{\text{global sections}} (T\times T)/W$

DAHA $\mathbb{H}_n(q,\tau)$

$\mathbb{C}[X_1^{\pm1},\dots,X_n^{\pm1}]\otimes_q \mathbb{C}[Y_1^{\pm1},\dots,Y_n^{\pm1}] \circlearrowright \mathbb{C}[X_1^{\pm1},\dots,X_n^{\pm1}]$

⇝ deformation quantization to difference operators

$X_i Y_j = q^{\delta_{ij}} Y_j X_i$

$Y_i \cdot f(X_1,\dots,X_i,\dots,X_n) = f(X_1,\dots,qX_i,\dots,X_n)$

DAHA: PBW basis

$$\underbrace{\underbrace{\mathbb{C}[X_1^{\pm1},\dots,X_n^{\pm1}]}_{\text{sub}}\otimes \underbrace{\mathcal{H}_{fin}}_{\text{sub}} \otimes \underbrace{\mathbb{C}[Y_1^{\pm1},\dots,Y_n^{\pm1}]}_{\text{sub}}}$$

$\mathcal{H}^X_{aff}$ $\qquad \mathcal{H}^Y_{aff}$

$\mathbb{C}[W^{aff}] = W\ltimes Q^\vee = W\ltimes \mathbb{Z}^r$

Dunkl presentation: $\mathcal{D}(T)$ $\mathcal{D}(t)$

$$\mathbb{H}_n(q,t) \longrightarrow \mathcal{D}_q(T)\#\mathbb{C}[W]$$
$$X_i \longmapsto x_i$$
$$T_i \longmapsto \tau s_i + (t-t^{-1})\frac{s_i-1}{X^{\alpha_i}-1}$$
$i=0,\dots,n-1$

Y_i's $\in \langle T_0, T_1,\dots,T_{n-1}\rangle$

all commute in $\mathcal{D}_q(T)\#\mathbb{C}[W]$

$\boxed{\text{DAHA}} \circlearrowright$ difference op's on T

trig DAHA $\circlearrowright$ differential operators on T

rat'l DAHA differential operators on

T $\quad W = S_n$

$W^{aff} = \langle T_0,\dots,T_n\rangle \cong W\ltimes \underbrace{\mathbb{Z}^r}_{Y_i}$

$Y_i = \underbrace{T_{i-1}^{-1}\cdots T_1^{-1} T_0}$

$(z_1,\dots,z_n)$ $\quad T_1: z_2 \leftrightarrow z_1$ $\quad T_0(z_1, \dots$ $= (qz_n \dots$

$$\mathbb{C}[\widetilde{B}_n(T^2)]/_{(T-\tau)(T+1)} \xrightarrow{\cong} \mathbb{H}_n(q,t)$$

Arthamonov–Shakirov

$\underline{Sk(\Sigma_2)} \circlearrowright Sk(H_2)$

$\mathbb{C}[B_n(\Sigma_g)]/_{(T-\tau)(T+1)}$ ← what sort of algebra do we get?

$Sk(\Sigma_g)$

$$\underline{\mathbb{H}} = \mathbb{C}[Y]\otimes \underbrace{\mathcal{H}_{fin}\otimes\mathbb{C}[X]}_{\mathfrak{b}\subseteq\mathfrak{g}} \qquad t \quad n$$

$Ch_G(\Sigma_g) = \{G\text{-local systems on } \Sigma_g\}$

$\mathrm{Dim}\, Ch_{GL_n}(T^2)$

$\boxed{n^2/G}$ $\quad 2n^2 \downarrow 4\ GL_n$

$\mathrm{Comm} \subset GL_n\times GL_n$

$\mathbb{C}[T] \otimes H \otimes \mathbb{C}[Y] \; \partial/\partial x_1 \cdots \partial/\partial x_n$

$- \; \mathbb{C}[T] \otimes \mathbb{C}[W] \otimes \mathbb{C}[t^\vee]$

$t \quad \mathbb{C}[t] \otimes \mathbb{C}[W] \otimes \mathbb{C}[t^\vee]$

quantum groups — $\mathcal{D}_q(G)$ – J — $V^{\otimes n} \otimes M$

Arakawa-Suzuki — $\mathfrak{g}((t))$ — $\mathcal{D}(G)$ Calaque-Enriquez-Etingof — $V^{\otimes} \otimes M$

$\mathfrak{g}[t]$ — $\mathcal{D}(\mathfrak{g})$ A-S — $V^{\otimes n} \otimes M$

$\ldots)$

$z_{n+1} := q z_1$

$z_{n+k} = q \cdot z_k$

$\ldots, z_n)$

$\ldots, z_2, \ldots, z_{n-1}, t z_1)$

Trig, Rat'l DAHA.

$\mathrm{DAHA} \underset{\mathbb{C}[q,t]}{\otimes} \mathbb{C}[[\hbar]]$

$t, c \in \mathbb{C}$.

q, τ

$$\left.\begin{array}{l} q = e^{t \cdot \hbar} \in \mathbb{C}[[\hbar]] \\ \tau = e^{c \cdot \hbar} \in \mathbb{C}[[\hbar]] \end{array}\right\} \quad q = e^{t\hbar} \quad I = \left\langle \begin{array}{c} Y_i - 1 \\ T_i - T_i^{-1} \end{array} \right\rangle$$

complete at that ideal.

$$\left.\begin{array}{l} \Rightarrow Y_i = e^{\hbar y_i} \\ \Rightarrow T_j = s_j e^{\hbar c s_j} \end{array}\right\} \quad y_i = \frac{1}{\hbar} \cdot \log(Y_i)$$

$$T_j^2 = (\tau - 1) T_j + 1$$

$$s_j^2 = \cancel{\hbar \cdot T_j} + 1 \quad + O(\hbar)$$

$$T_j = s_j \cdot \left(1 + \hbar c s_j + \frac{\hbar^2 c^2}{2} s_j^2 + \cdots\right)$$

Trig DAHA $:= \langle y_i, s_j, X_i \rangle \subseteq \mathrm{DAHA} \underset{\mathbb{C}(q,t)}{\otimes} \mathbb{C}((\hbar))$

$\mathbb{C}[W]$

Rat'l DAHA $\quad X_i = e^{\hbar x_i}$ $\quad \hbar = 0$.