Towards a functor between affine and finite Hecke categories in type A

Kostiantyn Tolmachov
partly joint with Roman Bezrukavnikov

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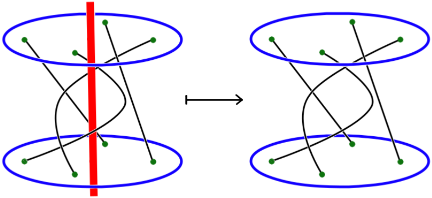
$$\mathbb{C}^* \hookrightarrow \mathbb{C}$$

$$\rightsquigarrow$$

$$B_n^{\mathrm{aff}} \to B_n$$



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Generators: $\{t_s, s \in I\}$; $t_i := t_{s_i}$.
Relations:

1. $t_i t_{i+1} t_i = t_{i+1} t_i t_{i+1}$.
2. $t_i t_j = t_j t_i, |i-j| > 1$.
3. $t_i^2 = 1 + (v^{-1} - v) t_i$.

$\mathbb{H}_n$ has a basis $\{t_w, w \in W\}$, defined by $t_w = t_{s_1} \dots t_{s_k}$ for a reduced expression $w = s_1 \dots s_k$.

Extended affine Hecke algebra

$(\mathbb{X}^*, \Phi, \mathbb{X}_*, \Phi^\vee)$ – root datum of GL_n.
$\mathbb{X}_* = \mathbb{X}^* =: \mathbb{X} \simeq \mathbb{Z}^n = \mathrm{span}_{\mathbb{Z}}\{e_1, \ldots, e_n\}$,
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4. $\theta_x \theta_y = \theta_{x+y}$.
5. $t_i \theta_j = \theta_j t_i$ if $j \neq i, i+1$.
6. $t_i \theta_i t_i = \theta_{i+1}$.
7. $\theta_0 = 1$.

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$\lambda \in \mathbb{X} \rightsquigarrow W\lambda$ – W-orbit.

Theorem (Bernstein)

The center of $\tilde{\mathbb{H}}_n^{aff}$ is a free $\mathbb{Z}[v, v^{-1}]$-module with a basis given by elements $\{z_\lambda, \lambda \in \mathbb{X}^+\}$,

$$z_\lambda := \sum_{\mu \in W\lambda} \theta_\mu.$$

Theorem (Dipper-James, Francis-Graham)

Set of symmetric polynomials in $\{JM_i\}$ is the center of $\mathbb{H}_n$.

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$Y \times Y$ is a $T \times T$-torsor over $\mathcal{B} \times \mathcal{B}$.
$\hat{D}_{\mathrm{fin}} := \hat{D}^b_{c,G,mon}(Y \times Y)$ – finite Hecke category.
$\hat{D}^b{}_{c,G,mon}(Y \times Y)$ – completed monodromic (with unipotent monodromy) bounded G-equivariant derived category of constructible sheaves on $Y \times Y$.

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$\mathfrak{hc} = D^b_G(G) \to D^b_G(Y \times Y), \mathfrak{hc} = a_! \pi^*[\dim Y]$ – Harish-Chandra functor.
$\hat{D}^b(\mathcal{CS})$ – completed derived category of character sheaves,
$\mathfrak{hc}(\hat{D}^b(\mathcal{CS})) \subset \hat{D}_{\mathrm{fin}}$.

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$V_\lambda \in \mathrm{Rep}(\mathrm{GL}_n) \rightsquigarrow \sum_\mu d(\lambda,\mu) z_\mu \in \tilde{\mathbb{H}}^{\mathrm{aff}}_n$, $d(\lambda,\mu)$ – weight multiplicities.

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$\Delta_*(V_\lambda \otimes \mathcal{O}_{\tilde{\mathfrak{g}}}) \rightsquigarrow s_\lambda \in \tilde{\mathbb{H}}_n^{\mathrm{aff}}$

$\Delta_* \pi_{\mathcal{B}}^* \mathcal{O}_{\mathcal{B}}(\mu) \rightsquigarrow \theta_\mu \in \tilde{\mathbb{H}}_n^{\mathrm{aff}}$.

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$$\mathsf{Rep}(\mathrm{GL}_n) \xrightarrow{B\circ\Psi\circ Sat} D^b(Coh_G(\hat{\mathrm{St}}))$$

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Conjecture

There are monoidal functors replacing the dashed arrows and categorifying (1).

Main result

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Theorem

There are monoidal functors ϖ_Z, ϖ, making the following diagram commutative and categorifying a part of diagram (1).

$$\begin{array}{ccc} \mathrm{Rep}(GL_n) & \xrightarrow{(B\circ\Psi\circ Sat)'} & D^G_{perf}(\hat{St}) \\ \varpi_Z \downarrow & & \downarrow \varpi \\ \hat{D}^b(\mathcal{CS}) & \xrightarrow{\mathfrak{hc}'} & \hat{D}_{fin} \end{array}$$

Functors from $\mathrm{Rep}(\mathrm{GL}_n)$

$\mathcal{C}$ – $\mathbb{C}$-linear, symmetric monoidal pseudo-abelian category.

$X \in \mathcal{C}, k \in \mathbb{Z}^+ \rightsquigarrow S_k$-action on X^k.

Partition λ of $k \rightsquigarrow S^\lambda X \in \mathcal{C}$ – Schur functors.

Lemma

Let $X \in Ob(\mathcal{C})$ be such that $\wedge^n X$ is invertible and $\wedge^{n+1} X = 0$. Then there is a monoidal functor

$$\mathrm{Rep}(GL_n) \to \mathcal{C}$$

sending the standard n-dimensional representation of $\mathrm{Rep}(GL_n)$ to X.

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$\mu \circ \xi = \mu \circ \tau$.

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$\mu \circ \xi = \mu \circ \tau$.
$A, B \in D^b_G(G) \rightsquigarrow A \boxtimes B \simeq \xi_! (A \boxtimes B)$,

$$A * B = \mu_!(A \boxtimes B) \simeq \mu_! \xi_! (A \boxtimes B) = \mu_! \tau_! (A \boxtimes B) \simeq B * A$$

– braided structure, $\beta_{A,B} : A * B \to B * A$.

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$A, B \in D^b_G(G) \rightsquigarrow A \boxtimes B \simeq \xi_!(A \boxtimes B)$,

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Lemma

If centralizers of all points in G are connected, then $D^b_G(G)$ is symmetric: $\beta_{A,B} \circ \beta_{B,A} = Id_{B*A}$

Braided structure on $D^b_G(G)$

G – any algebraic group. $\mu : G \times G \to G$ – multiplication map.
$A * B = \mu_!(A \boxtimes B)$ – monoidal structure on $D^b(G)$ (convolution).
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Corollary

For $G = GL_n$, $D^b_G(G)$ is a symmetric monoidal category.

Pro-units

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Exterior powers of the parabolic Springer sheaf

Theorem (Bezrukavnikov, T.)

1. $\wedge^k Spr_{P_0} \simeq (IC_{\lambda_k} \oplus IC_{\lambda_{k+1}})$, *for* $0 < k < n$.
2. $\wedge^n Spr_{P_0} \simeq IC_{\lambda_n}$
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Corollary

There is a monoidal functor $\varpi_Z : \mathrm{Rep}(GL_n) \to \hat{D}^b(\mathcal{CS})$, *sending a standard n-dimensional representation* V *to* $Spr_{P_0} * \hat{\delta}$, *and a monoidal functor*

$$\mathfrak{hc} \circ \varpi_Z : \mathrm{Rep}(G) \to \hat{D}_{fin},$$

sending representations of GL_n *to central objects.*

Lie algebra version

$\mathfrak{g} = \mathsf{Lie}(\mathrm{GL}_n)$.

$\mathrm{FT} : D^b_{\mathbb{G}_m}(\mathfrak{g}) \to D^b_{\mathbb{G}_m}(\mathfrak{g})$ – Fourier transform.

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Recall $\Psi \circ Sat : \mathrm{Rep}(\mathrm{GL}_n) \to \mathcal{P}_{G(\mathcal{O})}(Gr) \to \hat{D}_{\mathrm{aff}}$ – Gaitsgory's nearby cycles construction.

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Compare: V – standard representation of $G = GL(V)$,
$\Pi(s_V) = \Pi(\sum \theta_i) = \sum JM_i = t_e + t^2_{s_1} + \cdots + t_{s_{n-1} \ldots s_1} t_{s_1 \ldots s_{n-1}}$.

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Monodromy along $\mathbb{G}_m \subset \mathbb{G}_m \times \mathrm{GL}_{n-1}$ is an endomorphism of $\hat{\delta}_L$.

Induces an endomorphism of $Spr_{P_0} * \hat{\delta}$, which corresponds to m_V under ϖ.