Higgs Bundles and Global Springer Theory

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Plan of talk:

1. Ngô (2008)
2. Yun (2011)
3. Oblomkov–Yun (2014)

Sources:

- Ngô, « Le Lemme fondamental ... »
- Yun, "Global Springer Theory"
- Yun, "Lectures on Springer Theories..."
- Oblomkov–Yun, "Geometric Representations..."

§1 Ngô (2008)

Let $G = \mathrm{SL}_n(\mathbf{C})$ and $\mathfrak{g} = \mathfrak{sl}_n(\mathbf{C})$.

Centralizer group scheme:

$$I = \{(\gamma, g) \in \mathfrak{g} \times G \mid \mathrm{Ad}(g)\gamma = \gamma\}$$

For any field F and $\gamma \in \mathfrak{g}(F)$, we say that:

- γ is *regular* iff $\dim I_\gamma$ is minimal. In this case, I_γ is commutative.
- γ is *regular semisimple* iff I_γ is a torus.

Let $\mathfrak{g}^{\mathrm{rs}} \subseteq \mathfrak{g}^{\mathrm{reg}} \subseteq \mathfrak{g}$ be the corresponding loci.

Let $\mathfrak{c} = \mathbf{A}^{n-1} /\!\!/ S_n \simeq \operatorname{Spec} \mathbf{C}[e_2, \ldots, e_n]$.

The Chevalley map

$$\chi : \mathfrak{g} \to \mathfrak{c}$$

sends a matrix γ to the tuple $a = (a_i)_{i=2}^n$ given by

$$\det(t - \gamma) = t^n + a_2 t^{n-2} + \cdots + a_{n-1} t + a_n.$$

Let $\mathfrak{c}^\circ$ be the locus where this polynomial is separable.

Lem $\chi|_{\mathfrak{g}^{\mathrm{reg}}} : \mathfrak{g}^{\mathrm{reg}} \to \mathfrak{c}$ is surjective.

Lem $\mathfrak{g}^{\mathrm{rs}} = \chi^{-1}(\mathfrak{c}^\circ)$.

Lem $I|_{\mathfrak{g}^{\mathrm{reg}}}$ descends to $\mathfrak{c}$: There's a smooth group scheme J over $\mathfrak{c}$ and

$$\chi^* J|_{\mathfrak{g}^{\mathrm{reg}}} \xrightarrow{\sim} I|_{\mathfrak{g}^{\mathrm{reg}}}.$$

It extends to a morphism $\chi^* J \to I$.

Explicitly, if $\gamma \in \mathfrak{g}(F)$ and $\chi(\gamma) = a$, then:

$$J_a = \left\{ f \in (F[t]/a(t))^\times \;\middle|\; \prod_{\substack{\lambda \in \mathbf{C} \\ a(\lambda) = 0}} f(\lambda) = 1 \right\}$$

and $J_a \to I_\gamma$ sends $f \mapsto f(\gamma)$.

Ex If $\mathfrak{g} = \mathfrak{sl}_2$, then $\chi \simeq \det : \mathfrak{sl}_2 \to \mathbf{A}^1$.

$J(\mathbf{C})$ is a family of $\mathbf{C}^\times$'s degenerating to $\mathbf{C}^+ \rtimes \{\pm 1\}$.

Since J is a commutative group scheme, $\mathbb{B}J$ forms a commutative group stack over $\mathfrak{c}$.

The fiberwise action

$$\chi^*\mathbb{B}J = \mathbb{B}(\chi^* J) \curvearrowright \mathbb{B}I \quad \text{over } \mathfrak{g}$$

descends to a fiberwise action

$$\mathbb{B}J \curvearrowright \chi_*\mathbb{B}I = [\mathfrak{g}/G] \quad \text{over } \mathfrak{c}.$$

It is simply transitive on the regular loci of the fibers.

The geometry of this action underlies the geometry of both affine Springer fibers and Hitchin fibers.

Interlude Suppose $H \curvearrowright X$ and $H \curvearrowright V$. Recall:

- An H-bundle $E \to X$ is *principal* iff it trivalizes over an fpqc cover of X.
- The *associated bundle* $V_E \to X$ is defined by

  $$V_E = (E \times V)/H$$

  as an fpqc quotient.

Principal H-bundles are classified by maps $X \to \mathbb{B}H$.

Ex Suppose $L \to X$ is a line bundle.

Its frame bundle $L^\times \to X$ is a principal $\mathbf{G}_m$-bundle such that $L = (\mathbf{A}^1)_{L^\times}$.

Suppose X is integral, separated, noetherian, and $\hat{\mathcal{O}}_{X,v} \simeq \mathbf{C}[\![x]\!]$ for all $v \in X(\mathbf{C})$.

An *L-twisted G-Higgs bundle* on X is a section of

$$[\mathfrak{g}/G]_{L^\times} \to X,$$

where $\mathbf{G}_m \curvearrowright [\mathfrak{g}/G]$ by scaling $\mathfrak{g}$. Equivalent to (E, θ) with:

- $E \to X$ a principal G-bundle.
- θ a global section of $\mathfrak{g}_E \otimes L \to X$.

Since $G = \mathrm{SL}_n$, this is equivalent via $V = (\mathbf{A}^n)_E$ to:

- $V \to X$ a rank-n vector bundle with $\underline{\det}(V)$ trivial.
- θ a traceless global section of $\underline{\mathrm{End}}(V) \otimes L$.

The map $\chi : \mathfrak{g} \to \mathfrak{c}$ sends:

$$\text{scaling action } \mathbf{G}_m \curvearrowright \mathfrak{g}$$
$$\Downarrow$$
$$\text{weighted action } \mathbf{G}_m \curvearrowright \mathfrak{c} = \operatorname{Spec} \mathbf{C}[e_i]_{i=2}^n$$

The weights are $c \cdot e_i = c^i e_i$.

So χ induces a *Hitchin morphism* $h : \mathcal{M} \to \mathcal{A}$, where

$$\mathcal{M} = \mathcal{M}_{X,L} = \mathrm{H}^0(X, [\mathfrak{g}/G]_{L^\times}),$$

$$\begin{aligned}\mathcal{A} = \mathcal{A}_{X,L} &= \mathrm{H}^0(X, \mathfrak{c}_{L^\times}) \\ &= \textstyle\bigoplus_{i=2}^n \mathrm{H}^0(X, L^{\otimes i}).\end{aligned}$$

Intuitively, $h(V, \theta)$ lists coefficients of $\det_L(t - \theta)$.

The fiberwise action $\mathbb{B}J \curvearrowright [\mathfrak{g}/G]$ over $\mathfrak{c}$ is equivariant with respect to the $\mathbf{G}_m$-actions.

Therefore, $\mathcal{P} \curvearrowright \mathcal{M}$ over $\mathcal{A}$, where

$$\mathcal{P} = \mathcal{P}_X \coloneqq \mathrm{H}^0(X, (\mathbb{B}J)_{L^\times})$$

is called the *(relative) Picard stack*.

Motivation If X is a nice curve and $a \in \mathcal{A}$ is also nice, then:

- $\mathcal{P}_a$ parametrizes line bundles of a fixed degree on a certain branched cover $X_a \to X$.
- $\mathcal{M}_a$ is a certain compactification of $\mathcal{P}_a$.

We say that X_a is the *spectral curve* of a.

Global Picture Let X be a smooth proper curve. Fix $a = (a_i)_{i=2}^n \in \mathcal{A} = \bigoplus_{i=2}^n \mathrm{H}^0(L^{\otimes i})$.

Let y be a vertical coordinate on L, and let

$$X_a = \{y^n + a_2 y^{n-2} + \cdots + a_{n-1} y + a_n = 0\} \subseteq L.$$

Let $\mathcal{A}^\spadesuit$, *resp.* $\mathcal{A}^\heartsuit$, be the locus in $\mathcal{A}$ where X_a is integral, *resp.* reduced.

Lem If $a \in \mathcal{A}^\spadesuit$, then $\mathcal{M}_a$ is proper.

Lem If X has genus zero and $a \in \mathcal{A}^\heartsuit$, then

$$\mathcal{P}_a \simeq \mathit{Pic}^d(X_a) \quad \text{and} \quad \mathcal{M}_a \simeq \overline{\mathit{Pic}}^d(X_a),$$

where $d = \binom{n}{2} \deg L$.

Local Picture For all $v \in X(\mathbf{C})$, let

$$\hat{X}_v = \operatorname{Spec} \hat{\mathcal{O}}_v \quad \text{and} \quad \hat{X}_v^\circ = \operatorname{Spec} \hat{F}_v.$$

Abbreviate $a_v = a|_{\hat{X}_v}$ and $L_v = L|_{\hat{X}_v}$.

Prop If $a \in \mathcal{A}^\heartsuit(\mathbf{C})$ and $\gamma \in \chi^{-1}(a_v)$, then

$$[\mathcal{P}_{\hat{X}_v, a_v} \backslash \mathcal{M}_{\hat{X}_v, \hat{\mathcal{O}}_v, a_v}] \simeq [\mathcal{P}_\gamma \backslash \mathcal{M}_\gamma]$$

where

$$\mathcal{M}_\gamma = \{g \in G(\hat{F}_v)/G(\hat{\mathcal{O}}_v) \mid \operatorname{Ad}(g^{-1})\gamma \in \mathfrak{g}_{L^\times}(\hat{\mathcal{O}}_v)\},$$
$$\mathcal{P}_\gamma = I_\gamma(\hat{F}_v)/J_{a_v}(\hat{\mathcal{O}}_v),$$

given the structure of $\mathbf{C}$-ind-schemes.

Note: $\mathcal{M}_\gamma$ is a *(spherical)* affine Springer fiber.

Proof sketch

The fpqc quotient $G(\hat{F}_v)/G(\hat{\mathcal{O}}_v)$ classifies (E, ι) with:

- $E \to \hat{X}_v$ a principal G-bundle.
- $\iota : E|_{\hat{X}_v^\circ} \xrightarrow{\sim} G \times \hat{X}_v^\circ$.

$\mathcal{M}_\gamma$ classifies (E, θ, ι) with:

- $(E, \theta) \in \mathcal{M}_{\hat{X}_v, \hat{\mathcal{O}}_v, a_v}$.
- $\iota : E|_{\hat{X}_v^\circ} \xrightarrow{\sim} G \times \hat{X}_v^\circ$ such that $\iota(\theta) = \gamma$.

$\mathcal{P}_\gamma$ classifies (E', ι') with:

- $E' \to \hat{X}_v$ a principal J_{a_v}-bundle.
- $\iota' : E'|_{\hat{X}_v^\circ} \xrightarrow{\sim} I_\gamma \times \hat{X}_v^\circ$.

Local to Global Suppose L admits a square root. It defines a *Kostant section*

$$\mathfrak{c}_{L^\times} \to [\mathfrak{g}^{\mathrm{reg}}/G]_{L^\times},$$

which in turn induces a gluing map

$$\prod_{a(v) \notin \mathfrak{c}^\circ_{L^\times}} \mathcal{M}_{\gamma_v} \to \mathcal{M}_{X,L,a}$$

for any $a \in \mathcal{A}^\heartsuit(\mathbf{C})$ and $\gamma_v \in \chi^{-1}(a_v)$.

Thm (Ngô) If $a \in \mathcal{A}^\spadesuit(\mathbf{C})$, then any square root of L induces an algebraic homeomorphism

$$\frac{\mathcal{P}_{X,a} \times \prod_{a(v) \notin \mathfrak{c}^\circ_{L^\times}} \mathcal{M}_{\gamma_v}}{\prod_{a(v) \notin \mathfrak{c}^\circ_{L^\times}} \mathcal{P}_{\gamma_v}} \xrightarrow{\approx} \mathcal{M}_{X,L,a}.$$

Ex Let $G = \mathrm{SL}_2$ and $X = \mathbf{P}^1$ and $L = \mathcal{O}(2)$. Then

$$\mathcal{A} = \mathrm{H}^0(X, L^{\otimes 2}) = \mathrm{H}^0(\mathbf{P}^1, \mathcal{O}(4)).$$

Fix a coordinate $[x : 1]$ on X. Spectral curves look like

$$X_a = \{y^2 + a(x) = 0\} \subseteq L,$$

where $\deg a(x) = 4$.

If $a(x) = x^3$, then

$$\begin{aligned} \mathcal{M}_a &= \overline{Pic}^1(X_a) \simeq X_a \times \mathbb{B}\mu_2, \\ \mathcal{P}_a &= Pic^1(X_a) \simeq \mathbf{G}_a, \\ \mathcal{M}_{\gamma_0} \times \mathcal{M}_{\gamma_\infty} &= \mathbf{P}^1 \times pt, \\ \mathcal{P}_{\gamma_0} \times \mathcal{P}_{\gamma_\infty} &= \mathbf{G}_a \times 1. \end{aligned}$$

Note: $\overline{Pic}^1(X_a) \simeq X_a \times \mathbb{B}\mu_2$ for general $a \in \mathcal{A}^\spadesuit(\mathbf{C})$.

§2 Yun (2011)

Z. Yun's global Springer action fits into a table:

point	Springer fibers
disk $\hat{X}_v$	affine Springer fibers $\mathcal{M}_{\gamma_v}$
compact surface X	*parabolic* Hitchin fibers $\widetilde{\mathcal{M}}_a$

Full statement involves a graded $\mathbf{C}$-algebra

$$\mathbf{D}^{trig} = \mathrm{Sym}(\mathbf{V}_{\mathrm{KM}} \oplus \mathbf{C}) \otimes \mathbf{C}[W^{aff}].$$

By a Springer action, we really mean a morphism

$$\mathbf{D}^{trig} \to \bigoplus_i \mathrm{End}^{2i}(\tilde{h}_*^{\spadesuit}\mathbf{C}),$$

where $\tilde{h}^{\spadesuit}$ is a *parabolic* version of $h^{\spadesuit}$.

Here, $\mathbf{V}_{\mathrm{KM}} = \mathbf{X}^*(T_{\mathrm{KM}}) \otimes \mathbf{C}$, where

$$T_{\mathrm{KM}} = \mathbf{G}_m^{\mathrm{cen}} \times T \times \mathbf{G}_m^{\mathrm{rot}}$$

is the maximal torus of a certain Kac–Moody group

$$G_{\mathrm{KM}} = \widehat{LG} \rtimes \mathbf{G}_m^{\mathrm{rot}}.$$

Explicitly:

- $T \subseteq G$ is a maximal torus.
- LG is the *loop group* given by $LG(\mathbf{C}) = G(\mathbf{C}(\!(x)\!))$ on points, and

  $$1 \to \mathbf{G}_m^{\mathrm{cen}} \to \widehat{LG} \to LG \to 1$$

  is the central extension formed by the frame bundle of its determinant line bundle.
- $\mathbf{G}_m^{\mathrm{rot}}$ acts on LG and $\widehat{LG}$ by scaling x.

Fix a Borel $B \supseteq T$. Gives simple roots

$$\Delta = \{\alpha_1, \ldots, \alpha_r\} \subseteq \Phi^* \subseteq \mathbf{X}^*(T)$$

and affine simple roots

$$\Delta^{\mathit{aff}} = \{\alpha_0\} \cup \Delta \subseteq \mathbf{X}^*(T \times \mathbf{G}_m^{\mathrm{rot}}).$$

We have Weyl groups

$$W = \langle s_\alpha \rangle_{\alpha \in \Delta},$$
$$W^{\mathit{aff}} = \langle s_\alpha \rangle_{\alpha \in \Delta^{\mathit{aff}}} \simeq \mathbf{Z}\Phi_* \rtimes W.$$

Note: Since $G = \mathrm{SL}_n$, we have $\mathbf{Z}\Phi_* = \mathbf{X}_*(T)$.

We will use $W^{\mathit{aff}} \curvearrowright \mathbf{V}_{\mathrm{KM}}$ to define $\mathbf{D}^{\mathit{trig}}$.

Let u be an indeterminate.

The *trigonometric DAHA in the sense of Yun* is

$$\mathbf{D}^{\mathit{trig}} = \mathrm{Sym}(\mathbf{V}_{\mathrm{KM}} \oplus \mathbf{C}\langle u \rangle) \otimes \mathbf{C}[W^{\mathit{aff}}]$$

under this ring structure:

- $\mathbf{C}[W^{\mathit{aff}}]$ and $\mathrm{Sym}(\cdots)$ are subalgebras.
- u commutes with everything.
- For all $\xi \in \mathbf{V}_{\mathrm{KM}}$ and $\alpha \in \Delta^{\mathit{aff}}$, we have

$$s_\alpha \xi - {}^{s_\alpha}\xi s_\alpha = \langle \xi, \alpha^\vee \rangle u.$$

The grading is:

$$\deg w = 0 \qquad \text{for } w \in W^{\mathit{aff}},$$
$$\deg \xi = 2i \qquad \text{for } \xi \in \mathrm{Sym}^i(\cdots).$$

Write $\mathbf{X}^*(\mathbf{G}_m^{\mathrm{rot}}) = \mathbf{Z}\delta_{\mathrm{rot}}$. For any $c \in \mathbf{C}$, we set

$$\mathbf{D}_c^{trig} = \mathbf{D}^{trig}/(u + c\delta_{\mathrm{rot}}).$$

Still graded!

Rem The usual trig DAHA is $\mathbf{D}_c^{trig}/(\delta_{\mathrm{rot}} - 1)$ (up to sign??). Filtered, not graded!

Rem The subalgebra of $\mathbf{D}^{trig}$ or $\mathbf{D}_c^{trig}$ generated by

$$\mathrm{Sym}(\mathbf{V} \oplus \mathbf{C}\langle u \rangle) \otimes \mathbf{C}[W],$$

where $\mathbf{V} = \mathbf{X}^*(T) \otimes \mathbf{C}$, is Lusztig's graded AHA.

To get the W-part of the global Springer action, we must extend the Hitchin morphism h.

Let $f : \tilde{\mathfrak{g}} \to \mathfrak{g}$ be the Springer morphism, and let the top square below be cartesian:

$$\begin{array}{ccc} \widetilde{\mathcal{M}} & \longrightarrow & [\tilde{\mathfrak{g}}/G]_{L^\times} \\ \downarrow & & \downarrow f \\ \mathcal{M} \times X & \xrightarrow{eval} & [\mathfrak{g}/G]_{L^\times} \\ h \times \mathrm{id} \downarrow & & \downarrow \chi \\ \mathcal{A} \times X & \xrightarrow{eval} & \mathfrak{c}_{L^\times} \end{array}$$

Note that $[\tilde{\mathfrak{g}}/G] \simeq [\mathfrak{b}/B]$, where $\mathfrak{b} = \mathrm{Lie}(B)$.

Let $\tilde{h} : \widetilde{\mathcal{M}} \to \mathcal{A} \times X$ be the composition.

To construct

$$(\circledast) \qquad \mathbf{D}^{trig} \to \bigoplus_i \mathrm{End}^{2i}(\tilde{h}^{\spadesuit}_* \mathbf{C}),$$

we need to describe the actions of

- $w \in W^{aff}$.
- $\xi \in \mathbf{X}^*(\underbrace{\mathbf{G}_m^{\mathrm{cen}} \times T \times \mathbf{G}_m^{\mathrm{rot}}}_{T_{\mathrm{KM}}}) \oplus \mathbf{Z}u$.

The W^{aff}-action is built up via induction on s_α's, just like in affine Springer theory.

As for the lattice, we'll construct a map

$$\tilde{L} : \mathbf{X}^*(T_{\mathrm{KM}}) \oplus \mathbf{Z}u \to Pic(\widetilde{\mathcal{M}}),$$

then let $\xi \curvearrowright \tilde{h}^{\spadesuit}_* \mathbf{C}$ via cupping with $\tilde{h}^{\spadesuit}_* \mathrm{c}_1(\tilde{L}(\xi))$.

Let $\mathrm{Bun}_G^B = (\mathrm{Bun}_G \times X) \times_{\mathbb{B}G} \mathbb{B}B$. In each case,

$$\tilde{L}(\xi) = K|_{\widetilde{\mathcal{M}}}$$

for some map $\widetilde{\mathcal{M}} \to \mathrm{Bun}_G^B \to Z$ and $K \in Pic(Z)$.

Write $\mathbf{X}^*(\mathbf{G}_m^{\mathrm{rot}}) = \mathbf{Z}\delta_{\mathrm{rot}}$ and $\mathbf{X}^*(\mathbf{G}_m^{\mathrm{cen}}) = \mathbf{Z}\delta_{\mathrm{cen}}$.

ξ	Z	K
$\xi \in \mathbf{X}^*(T)$	$\mathbb{B}B$	$K(\xi)$
δ_{rot}	X	ω_X
δ_{cen}	Bun_G	ω_{Bun_G}
u	X	L

Above, $\xi \mapsto K(\xi)$ under $\mathbf{X}^*(T) \xrightarrow{\sim} Pic(\mathbb{B}B)$.

Thm (Yun) $(\circledast)$ is well-defined for $\deg(L) \geq 2g_X$.

(This condition ensures Ngô's "support theorem".)

Rem ($\circledast$) descends to $\mathbf{D}^{trig}_c = \mathbf{D}^{trig}/(u + c\delta_{\mathrm{rot}})$ iff

$$L \otimes \omega_X^{\otimes c} = \mathcal{O}_X.$$

This forces $c = -\deg(L)/(2g_X - 2)$.

Rem For all $(a, v) \in \mathcal{A}^{\spadesuit} \times X$, we get

$$\mathbf{D}^{trig} \curvearrowright \mathrm{H}^*(\widetilde{\mathcal{M}}_{a,v}, \mathbf{C})$$

by pullback and base-change.

But since ω_X and L trivialize upon pullback to v, the action factors through $\mathbf{D}^{trig}/(\delta_{\mathrm{rot}}, u)$.

To get interesting actions on fibers, need *orbifold* X and *equivariant* cohomology.

§3 Oblomkov–Yun (2014)

Let $\mathbf{G}_m^{(m)} \curvearrowright \mathbf{A}^2$ with weights $(m, 1)$. Then

$$X_m := [(\mathbf{A}^2 - (0,0))/\mathbf{G}_m^{(m)}]$$

is a *weighted projective line* in which ∞ has stabilizer μ_m and no other points are stacky.

Simultaneously,

- $\mathbf{G}_m^{\mathrm{rot}} \curvearrowright X_m$ via $t \cdot [x : z] = [tx : z]$.
- $\mathbf{G}_m^{\mathrm{dil}} \curvearrowright \mathfrak{g}, \mathfrak{c}$ and χ is $\mathbf{G}_m^{\mathrm{dil}}$-equivariant.

So for any $L \in Pic(X_m) \simeq \frac{1}{m}\mathbf{Z}$, we have

$$\mathbf{G}_m^{\mathrm{rot}} \times \mathbf{G}_m^{\mathrm{dil}} \curvearrowright \mathcal{M}_{X_m,L}, \widetilde{\mathcal{M}}_{X_m,L}, \mathcal{A}_{X_m,L}$$

and $\tilde{h} : \widetilde{\mathcal{M}} \to \mathcal{A}$ is equivariant.

Fix $c = d/m$ in lowest terms. Define $\mathbf{G}_m(c)$ as the subtorus acting on $a = (a_i)_i \in \mathcal{A}$ by

$$t^d \cdot a_i(x:z) = a_i(t^m x : z)$$

The points of

$$\mathcal{A}_c := \mathcal{A}^{\mathbf{G}_m(c)} = \mathbf{C}\langle x^{ic} z^{i(\deg(L)-c)m} \rangle_{i=2}^n$$

are said to be *homogeneous of slope c*.

Thm (OY) There are graded actions

$$\mathbf{D}^{trig} \to \mathrm{End}^{2*}_{\mathbf{G}_m^{\mathrm{rot}} \times \mathbf{G}_m^{\mathrm{dil}}}(\tilde{h}_!^{\heartsuit} \mathbf{C}),$$
$$\mathbf{D}_c^{trig} \to \mathrm{End}^{2*}_{\mathbf{G}_m(c)}(\tilde{h}_{c,!}^{\heartsuit} \mathbf{C}),$$

where $\tilde{h}_!^{\heartsuit}\mathbf{C}$, $\tilde{h}_{c,!}^{\heartsuit}\mathbf{C}$ are viewed as ind-complexes.

Cor $\mathbf{D}_c^{trig} \curvearrowright \mathrm{H}^*_{\mathbf{G}_m(c)}(\widetilde{\mathcal{M}}_{a,0})$ for $a \in \mathcal{A}_c(\mathbf{C})$.

There's also a *rational* degeneration of this story.

The *rational DAHA in the sense of Yun* is

$$\mathbf{D}^{rat} = \mathrm{Sym}(\mathbf{V} \oplus \mathbf{V}^\vee \oplus \mathbf{C}\langle u, \delta_{\mathrm{rot}} \rangle) \otimes \mathbf{C}[W]$$

under a graded ring structure we won't state. Let $\mathbf{D}_c^{rat} = \mathbf{D}^{rat}/(u + c\delta_{\mathrm{rot}})$.

Thm (OY) If $m = n$, the Coxeter number, then:

- $\mathcal{A}_c^{\heartsuit} = \mathcal{A}_c^{\spadesuit}$.
- There's a graded action

$$\mathbf{D}_c^{rat} \to \mathrm{End}^{2*}_{\mathbf{G}_m(c)}(\mathrm{gr}^{\mathrm{P}}_* \, \tilde{h}_{c,*}^{\heartsuit} \mathbf{C}),$$

where $\mathrm{P}_{\leq *}$ is the *perverse filtration* on $\tilde{h}_{c,*}^{\heartsuit}\mathbf{C}$.

Cor In this case, $\mathbf{D}_c^{rat} \curvearrowright \mathrm{gr}^{\mathrm{P}}_* \, \mathrm{H}^*_{\mathbf{G}_m(c)}(\widetilde{\mathcal{M}}_{a,0})$.

Ex Take $a = (0, \ldots, 0, x^d) \in \mathcal{A}_c(\mathbf{C})$, where d is coprime to n.

Here, $\mathcal{M}_{a,0} \simeq \overline{Pic}^{d(n-1)/2}(\{y^n + x^d = 0\})$ and $\widetilde{\mathcal{M}}_{a,0}$ is a "flagged" version.

Oblomokov–Yun:

$$\mathbf{D}_c^{trig} \curvearrowright \mathrm{H}^*_{\mathbf{G}_m(c)}(\widetilde{\mathcal{M}}_{a,0}),$$
$$\mathbf{D}_c^{rat} \curvearrowright \mathrm{gr}^{\mathrm{P}}_* \mathrm{H}^*_{\mathbf{G}_m(c)}(\widetilde{\mathcal{M}}_{a,0}).$$

If we specialize $\delta_{\mathrm{rot}} \to 1$ in the latter, then we get the *spherical simple module* of the usual rDAHA!

Garner–Kivinen have an alternate construction that does not rely on the perverse filtration.

Thank you for listening.