Affine Springer fiber – sheaf correspondence (Joint w/ Gorsky & Oblomkov)

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§ 1. Coulomb branches + Springer action

$G/\mathbb{C}$ reductive $\quad N \in \mathrm{Rep}(G) \xrightarrow{\text{BFN}} \mathcal{M}_c(G,N)$ affine variety

$N = \mathrm{Ad} = \mathfrak{g} = \mathrm{Lie}(G) \qquad \mathcal{M}_c(G,N) = \mathrm{Spec}\, H_*^{G_\mathcal{O}}(\mathcal{R}_{G,N})$

Upgrade: $\quad 1 \to G \to \tilde{G} \to G_F \to 1 \qquad G_F$ diagonalizable

(Think $\tilde{G} = G \times \mathbb{C}_m$, $\mathbb{C}_m \circlearrowright N$ by dilations, i.e. $N \in \mathrm{Rep}(\tilde{G})$)

$$\mathcal{M}_c(G,N) = \mathcal{M}_c(\tilde{G},N)/\!\!/G_F^\vee \, .$$

Pick $\chi \in X_*(G_F) \cong X^*(G_F^\vee)$

$\rightsquigarrow$ GIT quotient $\quad \mathcal{M}_c^\chi(G,N) \longrightarrow \mathcal{M}_c(G,N)$

In good cases, "symplectic resolution"

Thm (Bezrukavnikov – Finkelberg – Mirković)

$$T^*T^\vee/W = \mathcal{M}_c(G,\mathrm{Ad})$$

$W \circlearrowright T^\vee \qquad$ fixed points $\longleftrightarrow$ nodes in affine Dynkin diagram of G.

$S_2 \circlearrowright \mathbb{C}^\times \quad x \longmapsto x^{-1} \quad$ fps are $\{\pm 1\}$

locally around these fps, this is $\mathfrak{t} \oplus \mathfrak{t}^*/W$, where $\mathfrak{t}$ is a Cartan in Lie algebra given by removing corr. node.

BFN: **Def.** $\quad \mathcal{R}_{G,N} = \{ [g,s] \in G_\mathcal{K} \overset{G_\mathcal{O}}{\times} N_\mathcal{O} \mid gs \in N_\mathcal{O} \} \subset \mathcal{T}_{G,N} := G_\mathcal{K} \overset{G_\mathcal{O}}{\times} N_\mathcal{O}$

$\left(\mathcal{O} = \mathbb{C}[[t]] \right.$
$\left. \mathcal{K} = \mathbb{C}((t)) \right)$

Recall $\quad \mathrm{Gr}_G = G(\mathcal{K})/G(\mathcal{O}) \qquad H_*(\Omega G) \otimes H_*(\Omega G) \longrightarrow H_*(\Omega G)$

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$$H_*^{G_\mathcal{O}}(Gr_G) \otimes H_*^{G_\mathcal{O}}(Gr_G) \longrightarrow H_*^{G_\mathcal{O}}(Gr_G)$$ commutative, associative

$R_{G,N}$ inherits this multiplication:

$$\begin{tikzcd}
R_{G,N} \times R_{G,N} \arrow[d, hook] & p^{-1}(R_{G,N} \times R_{G,N}) \arrow[l, "p"'] \arrow[d, hook] \arrow[r, "q = /G_\mathcal{O}"] & q p^{-1}(R_{G,N} \times R_{G,N}) \arrow[r, "m"] & R_{G,N} \\
T_{G,N} \times R_{G,N} \arrow[d] & G_\mathcal{K} \times R_{G,N} \arrow[l, "p"'] & & \\
Gr_G \times Gr_G & G_\mathcal{K} \times Gr_G \arrow[l] \arrow[r] & G_\mathcal{K} \overset{G_\mathcal{O}}{\times} Gr \arrow[r] & Gr
\end{tikzcd}$$

$$m_* q_* p^* : H_*^{G_\mathcal{O}}(R_{G,N}) \otimes H_*^{G_\mathcal{O}}(R_{G,N}) \longrightarrow H_*^{G_\mathcal{O}}(R_{G,N})$$

comm + ass. + unital ring $\qquad \operatorname{Spec} H_*^{G_\mathcal{O}}(R_{G,N}) = \mathcal{M}_c(G,N) \dashrightarrow T^*T^\vee/W$

Next goal: 1) upgrade this to $\mathcal{M}^x(G,N)$ 2) Sheaves?

When N is 0, $\mathcal{M}_c(G,0) = \{(g,x) \in G^\vee \times (\mathfrak{g}^\vee_{reg})^* \mid g^{-1}xg = x\}/G^\vee$ = Universal (or regular) centralizer

1) $x : \mathbb{G}_m \longrightarrow G_F$ gives other "lattices" $N_i \subset N_\mathcal{K}$, $N_i := x(t)^i \cdot N_\mathcal{O} \subset N_\mathcal{K}$

(Think $x : \mathbb{G}_m \xrightarrow{id} \mathbb{G}_m$, $N_i = t^i \cdot N_\mathcal{O} \subset N_\mathcal{K}$)

Def. $${}_iR_j = \{[g,s] \in G_\mathcal{K} \overset{G_\mathcal{O}}{\times} N_j \mid gs \in N_i\}$$

Convolution gives products

$$ {}_iA_j \otimes {}_jA_k \longrightarrow {}_iA_k, \quad \text{where } {}_iA_j = H_*^{G_\mathcal{O}}({}_iR_j) $$

Note: In this "commutative" case, ${}_iA_j$ only depends on $i-j$

$$ \rightsquigarrow \quad {}_iA_0 \otimes {}_jA_0 \longrightarrow {}_{i+j}A_0 $$

$$ \rightsquigarrow \quad \mathrm{Proj}\Big(\bigoplus_{i\geq 0} {}_iA_0\Big) =: \mathcal{M}_C^{\chi}(G,N) \longrightarrow \mathcal{M}_C(G,N) $$

<u>Example</u> $G = GL_n$, $N = Ad$, $\chi = id$, $G_F = \mathbb{G}_m$

$$ \mathcal{M}_C^{\chi}(G,N) \cong \mathrm{Hilb}^n(\mathbb{C}^\times \times \mathbb{C}) \longrightarrow T^*T^\vee/W $$

(BFN: Can add framing to get $\mathrm{Hilb}^n(\widetilde{\mathbb{C}^2/\Gamma}) \longrightarrow \mathrm{Sym}^n(\mathbb{C}^2/\Gamma)$)

§2. Springer action

Want: $H_*^{G_\mathcal{O}}(R_{G,N}) \otimes H_*(X) \longrightarrow H_*(X)$

Hilburn-Kamnitzer-Weekes
Garner-K.

"BFN Springer" / "generalized affine Springer" theory

$N = Ad$: $$ Sp_\gamma = \Big\{[g]\in Gr_G \;\Big|\; g\gamma g^{-1} \in \underbrace{\mathrm{Lie}\, G(\mathcal{O})}_{= N_\mathcal{O}}\Big\} $$

In general $\quad S_{p+i\gamma} = \{[g] \in Gr_G \mid g^{-1}\gamma g \in t^{-1}\mathrm{Lie}\,G(\mathcal{O})\}$

Key lemma: Let $\eta \subset N_K$ $G_{\mathcal{O}}$-stable lattice.

$$G_K \cdot \gamma \cap \eta = {}_\eta\mathcal{O}_\gamma \qquad \text{"orbital variety"}$$

$$\begin{tikzcd}
& {}_\eta X_\gamma = \{g \in G_K \mid g\gamma \in \eta\} \arrow[dl, "\mathrm{Stab}_{G_K}(\gamma) = L_\gamma"'] \arrow[dr, "G_{\mathcal{O}}"] & \\
{}_\eta\mathcal{O}_\gamma & & {}_\eta M_\gamma = \{g \in Gr_G \mid g\gamma \in \eta\}
\end{tikzcd}$$

If $\eta = t^i N_{\mathcal{O}}$, ${}_\eta M_\gamma = S_{p+i\gamma}$

In particular $\quad H_*^{L_\gamma}(\underbrace{{}_\eta M_\gamma}_{\text{(Think } S_{p\gamma})}) \cong H_*^{G_{\mathcal{O}}}({}_\eta\mathcal{O}_\gamma)$

Convolution

$$\cdot \; {}_jR_i \times {}_i\mathcal{O}_\gamma \longleftarrow p^{-1}({}_jR_i \times {}_i\mathcal{O}_\gamma) \longrightarrow q p^{-1}({}_jR_i \times {}_i\mathcal{O}_\gamma) \longrightarrow {}_j\mathcal{O}_\gamma$$

$$\rightsquigarrow \quad H_*^{G_{\mathcal{O}}}({}_jR_i) \otimes H_*^{L_\gamma}(S_{p+i\gamma}) \longrightarrow H_*^{L_\gamma}(S_{p+j\gamma})$$

Thm (GKO, BFM, BFN, Finkelberg–Tsymbaliuk, ...) $\quad$ (G reductive, $N = Ad$)

$\quad$ $\tilde{G}_{\mathcal{O}} \rtimes \mathbb{G}_m^{rot}(\ldots) \to H^{G_m \times G_m}(pt) = \mathbb{C}[\hbar, \epsilon]$

$${}_0A_0 \cong \mathbb{C}[T^*T^\vee]^W \quad , \quad {}_0A_0^\hbar := H_*^{G_{\mathcal{O}}\rtimes \mathbb{G}_m^{rot}}(\mathcal{R}_{G,N}) = H_*^{\dots}(\dots)$$

$$\cong e H_\hbar^{trig} e = \text{Spherical trig. Cherednik}$$

(There is a version with I = Iwahori in place of $G_{\mathcal{O}}$, s.t. ${}_0\mathcal{I}_0 = H_*^{I\rtimes \mathbb{G}_m^{rot}}(\widetilde{\mathcal{R}}_{G,N})$)

$$\cong H_\hbar^{trig}$$

+ The action ${}_{-j}A_{-i}^\hbar \otimes H_*^{H_\gamma}(S_{p+i,\gamma}) \longrightarrow H_*^{H_\gamma}(S_{p+j,\gamma})$ (Here $H_\gamma \subset L_\gamma \cap \widetilde{G}_k \rtimes \mathbb{G}_m^{rot}$)

generalizes Springer action from [OY], [VV]

<u>Remark</u> Works for any $\gamma \in \mathfrak{g}((t))^{r.s.}$ (quantization only OK for homogeneous γ)

(<u>Ex.</u> $T(\mathbb{C}) \subset T(K) = \mathrm{Stab}_{G(K)}\begin{pmatrix} a_1 t & & \\ & \ddots & \\ & & a_n t\end{pmatrix}$)

$\gamma \hookrightarrow$ Procesi (in earlier talks)

<u>Thm</u>

$$\left.\underbrace{\bigoplus_{i\le j\le 0} {}_iA_j}_{\mathbb{Z}\text{-algebra (Iain's talk)}} \circlearrowright \bigoplus_{i\ge 0} H_*^{H_\gamma}(S_{p+i,\gamma})\right\} \rightsquigarrow \text{q.c. sheaf on } \mathcal{M}^\chi(G,N)$$

${}_iA_j$ comes from $\bigoplus_{d\le 0} {}_dA_0$

What kind of sheaf? <u>Ex</u> γ slope $\frac{kn+1}{n}$ homogeneous $\left(\gamma = t\left(te_{\theta^\vee} + \sum_{\text{simple roots}} e_{\alpha^\vee}\right)\right)$

$G = GL_n$

The sheaf we get from the above construction, say $\mathcal{F}_\gamma$ is $\underline{\mathcal{O}(k)}\Big|_{\mathrm{Hilb}^n(\mathbb{C}^\times \dots)}$

$N = Ad$

Ample line bundle on $Hilb^n(\mathbb{C}^\times \times \mathbb{C})$ — $Hilb^n(\mathbb{C}^\times \times \mathbb{C})_{(1,0)}$

A way to show this is:
1) Compare $\bigoplus_i A_j^*$ to Gordon-Stafford
2) Use the fact that $H_*^{G_m}(Sp_\gamma) = e L_{\frac{kn+1}{n}}(\text{triv})$
3) Use Gordon-Stafford's results on the image of $e L_{\frac{kn+1}{n}}$ under $(\bigoplus_i A_j^k)$-grmod $\to Coh(Hilb^n(\mathbb{C}^2))$

Ex. Slope $k = \frac{kn}{n}$ homogeneous $\gamma = \text{diag}(a_1, \dots, a_n) \cdot t^k$

$\mathcal{F}_\gamma \cong P|_{Hilb^n(\mathbb{C}^\times \times \mathbb{C})}$ (To prove this, need Haiman's results)

$$H_*^T(Sp_\gamma) \cong \bigcap_{\alpha \in \Phi^+} \langle 1 - \alpha^\vee, y_\alpha \rangle^k \subset \mathbb{C}[T^* T^\vee]$$

$= I^{(k)}$ (To compare to J^k, need Haiman)

J = ideal generated by antisym polynomials in $\mathbb{C}[T^* T^\vee]$

§3. Commuting varieties

Def. $Comm_{G^\vee} = \{(g, x) \in G^\vee \times (\mathfrak{g}^\vee)^* \mid g x g^{-1} = x\} / G^\vee$

Thm (Losev, based on Joseph)

$$(Comm_{G^\vee})_{red} \cong T^* T^\vee / W$$

$= \mathcal{M}_C(G, Ad)$

(Map: $\mathbb{C}[G^\vee \times (\mathfrak{g}^\vee)^*]^{G^\vee} \xrightarrow{res} \mathbb{C}[T^* T^\vee]^W$)

Partial resolution: Thm (G anything)

$$e I^{(d)} \cong H_*^{G_\mathcal{O}}(d R_0)$$

$e = \frac{1}{|W|}\sum_{w\in W} w$ $\qquad \mathrm{Proj}\left(\bigoplus_{d\geq 0} e I^{(d)}\right) \longrightarrow T^*T^\vee/W$

Q. What does this look like?

In rational case, let $I_+ = \bigcap_{\alpha\in\Phi^+} \langle x_{\alpha^\vee}, y_\alpha \rangle \subset \mathbb{C}[\mathfrak{t}\oplus\mathfrak{t}^*]$

$\mathrm{Proj}\left(\bigoplus_d I_+^{(d)}\right) \longrightarrow \mathfrak{t}\oplus\mathfrak{t}^*/W$ Conical. Q: What are fixed points wrt $(\mathbb{C}^\times)^2$?

Type BC: Expect this to be $\mathbb{Z}/2\mathbb{Z}$ – $\mathrm{Hilb}^n(\mathbb{C}^2)$

General facts about $\mathcal{F}_\gamma$: If $\{1\} = H_\gamma \subseteq L_\gamma$ trivial

$\bigoplus_{i\geq 0} H_*(Sp_{\gamma + i}) \rightsquigarrow \mathcal{F}_\gamma$ supported on preimage of $\{0\}\times T^\vee/W$

If γ elliptic, $\mathcal{F}_\gamma$ is supported at the preimage of finitely many points in $T^*T^\vee/W$, and can use "endoscopic" decomposition (Ngô)

$$H_*(Sp_\gamma) = \bigoplus_{\kappa\in X_*(T)} H_*(Sp_\gamma)_\kappa$$ to write

Sheaf at each of these points as $\mathcal{F}_\gamma^H$, for $H\subset G^\vee$ endoscopic group

Lives on a different variety

If $H_\gamma = T(\mathbb{C})$ for γ split, $\bigoplus H_*^{T(\mathbb{C})}(S_{p \dots}) \rightsquigarrow \mathcal{F}$ supported everywhere (equivariantly formal) $i \geq 0$

(In general only know it's a subvariety, need to know more about action to say more)

Dimofte - Garner - Geracie - Hilburn : "3d Mirror symmetry"

$$\text{"Line operators"} \left\{ \text{D-mod}(N_K/G_K) \longleftrightarrow \mathrm{Coh}\left(\mathcal{M}_C^{\chi}(G,N)\right) \right.$$

$\gamma \rightsquigarrow$ Skyscraper sheaf here

$$\rotatebox{90}{$\in$} \; \tilde{\mathcal{F}}_\gamma$$